A15273W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

SUPERSYMMETRY AND SUPERGRAVITY

Trinity Term 2022

Thursday, 21st April 2022, 9:30am-11:30am

You should submit answers to both questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

You are permitted to use the following material(s):
Calculator (candidate to provide)
The use of computer algebra packages is **not** *allowed.*
A4 summary sheet

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. (a) [4 marks] Consider the $\mathcal{N}$-extended super-Poincaré algebra in 4d Minkowski spacetime. Write down the commutators between the Poincaré generators and the supercharges, and the anticommutators among supercharges.

(b) [2 marks] Consider 4d $\mathcal{N}=2$ supersymmetry. Describe the main differences between long and short massive supermultiplets.

(c) [6 marks] Consider minimal 4d $\mathcal{N}=1$ supersymmetry. Let $H=P^0$ denote the Hamiltonian operator. Prove that $\langle\psi|H|\psi\rangle \geqslant 0$ for any state $|\psi\rangle$. Prove that $H|\psi\rangle=0$ if and only if $Q_\alpha|\psi\rangle=0$ and $\overline{Q}_{\dot\alpha}|\psi\rangle=0$. Explain why this result is relevant in the study of spontaneous supersymmetry breaking.

Hint: In the conventions of the lectures, $\{Q_\alpha, \overline{Q}_{\dot\beta}\} = 2\begin{pmatrix} P^0+P^3 & P^1-i\,P^2 \\ P^1+i\,P^2 & P^0-P^3\end{pmatrix}_{\alpha\dot\beta}$.

(d) [13 marks] Let us consider supersymmetric quantum mechanics with one complex supercharge. We can regard quantum mechanics as a quantum field theory in (0+1)-dimensions. Thus, a field is a function of time t. In analogy with 4d $\mathcal{N}=1$ supersymmetry, one can introduce a notion of superspace. It is parametrized by time t and two Grassmann-odd coordinates θ, $\overline{\theta}$, satisfying the reality condition $\theta^*=\overline{\theta}$. Time translations and supersymmetry variations are implemented by differential operators. Let us define

$$\mathbf{Q} = i\left(\frac{\partial}{\partial\theta} - i\,\overline{\theta}\,\frac{\partial}{\partial t}\right), \qquad \overline{\mathbf{Q}} = i\left(\frac{\partial}{\partial\overline{\theta}} - i\,\theta\,\frac{\partial}{\partial t}\right), \qquad \mathbf{H} = i\,\frac{\partial}{\partial t}\,. \tag{1}$$

- [2 marks] Verify by explicit computation that

$$\{\mathbf{Q}, \overline{\mathbf{Q}}\} = 2\,\mathbf{H}\,. \tag{2}$$

Just like in 4d $\mathcal{N}=1$ superspace, one can define supersymmetry covariant derivatives, which anticommute with the differential operators $\mathbf{Q}$, $\overline{\mathbf{Q}}$. They are given by

$$D = \frac{\partial}{\partial\theta} + i\,\overline{\theta}\,\frac{\partial}{\partial t}\,, \qquad \overline{D} = \frac{\partial}{\partial\overline{\theta}} + i\,\theta\,\frac{\partial}{\partial t}\,. \tag{3}$$

A *real superfield* is a function of t, θ, $\overline{\theta}$ of the form

$$X(t,\theta,\overline{\theta}) = x(t) + \theta\,\psi(t) - \overline{\theta}\,\overline{\psi}(t) + \theta\,\overline{\theta}\,\mathcal{D}(t)\,, \tag{4}$$

where the component fields x, ψ, $\overline{\psi}$, $\mathcal{D}$ obey the reality conditions

$$x^* = x\,, \qquad \psi^* = \overline{\psi}\,, \qquad \mathcal{D}^* = \mathcal{D}\,. \tag{5}$$

The fields x, $\mathcal{D}$ are Grassmann-even, while ψ, $\overline{\psi}$ are Grassmann-odd.
We can construct supersymmetric Lagrangians by integrating a real superfield over θ, $\overline{\theta}$.

- [4 marks] Consider a real superfield X as in (4) and compute

$$L_{\text{kin}} = \int d\overline{\theta}\,d\theta\left(-\tfrac{1}{2}\,\overline{D}X\,DX\right)\,. \tag{6}$$

  Hint: For the integral over Grassmann-odd coordinates, use $\int d\overline{\theta}\,d\theta\,\theta\,\overline{\theta}=1$.

- [4 marks] Let h be a real analytic function. Let us consider the composite real superfield $h(X)$. Prove that

$$\int d\overline{\theta}\,d\theta\,h(X) = h'(x)\,\mathcal{D} - h''(x)\,\overline{\psi}\,\psi\,, \tag{7}$$

  where a prime denotes differentiation.

  Hint: The $\theta\overline{\theta}$ component of $h(X)$ can be extracted by computing the quantity $\frac{\partial}{\partial\overline{\theta}}\frac{\partial}{\partial\theta}h(X(t,\theta,\overline{\theta}))$, and setting $\theta=0=\overline{\theta}$ at the end of the computation.

- [3 marks] We may now consider the total Lagrangian

$$L_{\rm tot} = \int d\bar\theta \, d\theta \Big(-\tfrac{1}{2} \, \overline{D}X \, DX - h(X) \Big) \ . \tag{8}$$

  Write $L_{\rm tot}$ in terms of the component fields x, ψ, $\overline{\psi}$, $\mathcal{D}$. Integrate out the auxiliary field $\mathcal{D}$ and write the resulting Lagrangian in terms of x, ψ, $\overline{\psi}$.

2. (a) [2 marks] Define the notion of a chiral superfield in 4d $\mathcal{N}=1$ superspace. Introduce and define superspace differential operators as needed.

   (b) [2 marks] Write the most general renormalizable action in superspace for a model with a collection Φ^i $(i = 1, \dots, n)$ of chiral superfields (and no vector superfields).

   (c) [6 marks] Let us consider a renormalizable model with three chiral superfields X, Y, Z and superpotential

$$\text{(i)} \;\; W = g\, X\, Y\, Z \ , \qquad \text{or} \qquad \text{(ii)} \;\; W = \lambda\, X + m\, Y\, Z + g\, X\, Y^2 \ . \tag{9}$$

   Assume m, g, λ are generic, non-zero complex parameters. Determine the classical space of supersymmetric vacua of the model (i) and the model (ii).

   (d) [2 marks] Describe the field content of an off-shell 4d $\mathcal{N}=1$ vector multiplet for gauge group G. How do the the various fields transform under G?

   (e) [2 marks] Consider an Abelian vector superfield V and its field strength superfield $\mathcal{W}_\alpha = -\frac{1}{4}\, \overline{D}_{\dot\beta}\, \overline{D}^{\dot\beta}\, D_\alpha V$. Prove that $\mathcal{W}_\alpha$ is invariant under a gauge transformation

$$V \mapsto V + \tfrac{i}{2}\, (\Lambda - \overline{\Lambda}) \ , \tag{10}$$

   where Λ is a chiral superfield.

   (f) [11 marks] Let us consider supersymmetric quantum electrodynamics (SQED), i.e. a 4d $\mathcal{N}=1$ gauge theory with gauge group $U(1)$, one chiral superfield X^+ of charge $+1$, and one chiral superfield X^- of charge -1. (By a common abuse of notation, we shall use the same symbol for a chiral superfield, its scalar component, and the vacuum expectation value (VEV) of the latter, depending on context.) We consider the model without superpotential, but we turn on a non-zero Fayet-Iliopoulos parameter.

   - [1 mark] Explain why the classical space of supersymmetric vacua (moduli space) of the model is described by an equation of the form

$$|X^+|^2 - |X^-|^2 = \gamma \ , \tag{11}$$

     where γ is a non-zero real constant.
   - [2 marks] The moduli space of the model is the space of solutions to (11) modulo $U(1)$ gauge transformations. What is the real dimension of this space?
   - [2 marks] What happens to the $U(1)$ gauge symmetry on moduli space?
   - [6 marks] Let us define the gauge-invariant chiral superfield

$$M = X^+ \, X^- \ .$$

     The low-energy dynamics around a generic point in moduli space can be described by an effective theory for the field M, with an effective Kähler potential $K_{\rm eff}(M, M^\dagger)$. Compute $K_{\rm eff}(M, M^\dagger)$ at the classical level.

     Hint: The Kähler potential of SQED is known as a function of $X^\pm$, $(X^\pm)^\dagger$. On the moduli space, we can trade $X^\pm$, $(X^\pm)^\dagger$ for M, $M^\dagger$ and γ.