Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

SUPERSYMMETRY AND SUPERGRAVITY

Trinity Term 2020

THURSDAY, 4TH JUNE 2020, 14:30

You should submit answers to both of the two questions.

You have **3 hours** *to complete the paper and upload your answer file.*
You are permitted to use the following material(s):
Calculator
The use of computer algebra packages is **not** *allowed*

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

1. (a) [3 marks] Describe the $\mathcal{N}$-extended super-Poincaré algebra in any space-time dimension: List the bosonic and fermionic generators, and give the *schematic form* (i.e. without explicit indices or numerical factors) of the (anti)commutation relations. You may ignore the central charges.

   (b) [2 marks] What is the maximal $\mathcal{N}$ allowed in 4d for interacting field theories without gravity? And with gravity? Justify your answer.

   (c) [1 mark] Give all the anti-commutators amongst the 4d $\mathcal{N}=1$ supercharges Q_α and $\bar{Q}_{\dot\alpha}$, in Weyl spinor notation.

   (d) [3 marks] Briefly explain what is a 4d $\mathcal{N}=1$ superfield. Expand a general real scalar superfield $\mathcal{S}(x,\theta,\bar\theta)$ in terms of its field components, and count the number of (off-shell) bosonic and fermionic degrees of freedom.

   (e) [3 marks] The 4d $\mathcal{N}=1$ supercharges are realized on superspace as the differential operators:

$$\mathbf{Q}_\alpha = -i(\partial_\alpha - i(\sigma^\mu\bar\theta)_\alpha\partial_\mu)\ , \qquad \bar{\mathbf{Q}}_{\dot\alpha} = i(\bar\partial_{\dot\alpha} - i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu)\ . \tag{1}$$

   with $\partial_\alpha = \frac{\partial}{\partial\theta^\alpha}$, $\bar\partial_\alpha = \frac{\partial}{\partial\bar\theta^{\dot\alpha}}$ and $\partial_\mu = \frac{\partial}{\partial x^\mu}$. By explicit computation, show that they realise the supersymmetry algebra (only check the anti-commutators).

   (f) [3 marks] Define the notion of a chiral superfield, denoted by Φ. Introduce and define superspace differential operators as needed. Why is it called chiral?

   (g) [10 marks] Here, we will study minimal supersymmetry in 3d (with space-time signature $(-1,1,1)$). The two real supercharges of $ISO(1,2|2)$ sit in a single 3d Majorana spinor Q_α. Let $\mu,\nu,\dots \in \{0,1,2\}$ denote the space-time coordinate indices, and $\alpha,\beta,\dots\in\{1,2\}$ denote the spinor indices. The spinor indices are raised with $\varepsilon^{\alpha\beta}$ and $\varepsilon_{\alpha\beta}$. (The contraction conventions are like for undotted Weyl indices $\alpha,\beta,\cdots$ in 4d.) We pick the 3d γ-matrices:

$$(\gamma^\mu)_\alpha{}^\beta = \left\{\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\ ,\ \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\ ,\ \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\}\ . \tag{2}$$

   Then, the 3d $\mathcal{N}=1$ supersymmetry algebra contains the anti-commutators:

$$\{Q_\alpha, Q_\beta\} = \gamma^\mu_{\alpha\beta}P_\mu\ . \tag{3}$$

   (Hint: Useful identities are: $\theta^\alpha\theta^\beta = -\frac{1}{2}\theta\theta\varepsilon^{\alpha\beta}$ and $\gamma^\mu\gamma^\nu = \eta^{\mu\nu} - \epsilon^{\mu\nu\rho}\gamma_\rho$, with $\epsilon^{012}=1$. Note also that $\gamma^\mu_{\alpha\beta} = \gamma^\mu_{\beta\alpha}$.)

   - Derive the general structure of the massless one-particle supermultiplet (with energy $E\neq 0$).
   - Define 3d $\mathcal{N}=1$ superspace $\mathbb{R}^{1,2|2}$ as a coset manifold with coordinates (x^μ,θ^α), with θ^α some real Grassmann coordinates. Then, derive the induced action of the supercharge Q_α on superfields. You should find:

$$\mathbf{Q}_\alpha = -i\left(\frac{\partial}{\partial\theta^\alpha} - \frac{i}{2}(\gamma^\mu\theta)_\alpha\frac{\partial}{\partial x^\mu}\right)\ . \tag{4}$$

   - Consider the real 3d $\mathcal{N}=1$ superfield:

$$X(x,\theta) = \varphi(x) + \theta\psi(x) + \frac{1}{4}\theta\theta M(x)\ , \tag{5}$$

   with φ a real scalar, ψ_α a 3d fermion, and M an auxiliary real scalar. Using $\mathbf{Q}_\alpha$ above, derive the supersymmetry variations of the component fields.

- Prove that the 3d action:

$$S = \int d^3x \left(-\frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - i\psi\gamma^\mu \partial_\mu \psi - \frac{1}{2}M^2 \right) \tag{6}$$

is $\mathcal{N} = 1$ supersymmetric.

2. (a) [4 marks] Write down schematically the Lagrangian for the most general renormalizable 4d $\mathcal{N} = 1$ supersymmetric gauge theory, in superspace notation. Name the various terms, and briefly explain their physical content.

   (b) [2 marks] Briefly describe the general structure of the Wilsonian effective action of a theory of chiral multiplets only (no gauge fields), in superspace.

   (c) [6 marks] The Wess-Zumino model (for a single chiral multiplet) has the Lagrangian:

$$\begin{aligned} \mathcal{L} = &-\partial_\mu \bar{\phi} \partial^\mu \phi - m^2 \bar{\phi}\phi - i\bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi - \frac{m}{2}(\psi\psi + \bar{\psi}\bar{\psi}) \\ &- \lambda^2 \bar{\phi}^2 \phi^2 - m\lambda(\bar{\phi}^2\phi + \bar{\phi}\phi^2) - \lambda(\psi\psi\phi + \bar{\psi}\bar{\psi}\bar{\phi}) \ . \end{aligned} \tag{7}$$

   - Give the Feynman rules and write down all the one-loop diagrams that contribute to the self energy of the scalar, $\Pi_{\bar{\phi}\phi}(p^2)$.
   - Show that $\lim_{p^2 \to 0} \Pi_{\bar{\phi}\phi}(p^2) = 0$.
   - Briefly discuss the physical meaning of this result, including its possible relevance for particle physics.

   (d) [5 marks] Consider a Wess-Zumino model of three chiral superfields X, Y and Z, with the superpotential:

$$W = mXY + \lambda XYZ + \frac{g}{3}Z^3 \ , \tag{8}$$

   where $m, \lambda, g \in \mathbb{C}^*$ are non-zero coupling constants.

   - What is the global (internal) symmetry of this model?
   - Analyse the vacuum structure of the model. Does the model preserve supersymmetry? (Assume a canonical kinetic term.)
   - Study the limit $m = 0$. What are the global symmetries? How does the vacuum structure change?

   (e) [4 marks] Consider a theory of n chiral multiplets Φ_i. Using the general form of the classical Lagrangian (with canonical kinetic term), prove that, at the classical level, F-term supersymmetry breaking implies the existence of a massless fermion.

   (f) [4 marks] Let S^μ_α denote the conserved supersymmetry current of a QFT. Using the supersymmetry algebra, give the general structure of the anti-commutator:

$$\{Q_\alpha, \bar{S}^\mu_{\dot{\beta}}\} \ , \tag{9}$$

   in terms of bosonic operators in the theory. Discuss the schematic structure of the supercurrent multiplet $\mathcal{S}_\mu$, which contains the operator S^μ_α. What is the minimal number of independent bosonic and fermionic operators that it contains (assuming the theory is not conformally invariant)? Write down the minimal coupling of S^μ_α and of its superpartners to sources, at first order in the sources. What is the physical interpretation of the supersymmetry multiplet of sources?