Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

SUPERSYMMETRY AND SUPERGRAVITY

Trinity Term 2019

THURSDAY, 25TH APRIL 2019, 2.30pm to 4.30pm

You should submit answers to both questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. (a) [4 marks] Write down the $\mathcal{N}$-extended super-Poincaré algebra in four-dimensional Minkowski space-time, in Weyl spinor notation. (You may omit writing down the Poincaré algebra itself, but do include the commutators between the supercharges and the Poincaré generators.)

   (b) [1 mark] Prove that all the particles in a supermultiplet have the same energy.

   (c) [6 marks] Explain how to construct representations of the 4d $\mathcal{N}$-extended supersymmetry algebra on massless one-particle states, for any $\mathcal{N}$. Make sure that your final answer is CPT invariant. [*Pick the Lorentz frame* $p^\mu = (E, 0, 0, E)$ *for the energy-momentum of the one-particle states. Start your explanation with a discussion of how the supersymmetry algebra acts on those states.*]

   (d) [2 marks] As a special case of the general construction above, construct the unique *supergravity* multiplet of 4d $\mathcal{N} = 8$ supersymmetry. Describe the particle content. Why is the supermultiplet unique?

   (e) [1 mark] Give a definition of 4d $\mathcal{N} = 1$ superspace.

   (f) [5 marks] By explicit computation, show that the differential operators on superspace:

$$\begin{aligned} \mathbf{Q}_\alpha &= -i(\partial_\alpha - i(\sigma^\mu\bar\theta)_\alpha\partial_\mu) \ , \\ \bar{\mathbf{Q}}_{\dot\alpha} &= i(\bar\partial_{\dot\alpha} - i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu) \ , \\ \mathbf{P}_\mu &= -i\partial_\mu \ , \\ \mathbf{M}_{\mu\nu} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu - (\theta\sigma_{\mu\nu})^\alpha\partial_\alpha + (\bar\sigma_{\mu\nu}\bar\theta)^{\dot\alpha}\partial_{\dot\alpha}) \ , \end{aligned} \tag{1}$$

   satisfy the 4d $\mathcal{N} = 1$ super-Poincaré algebra. Only check the commutation relations involving at least one supercharge. The following identity may be useful:

$$\sigma^{\mu\nu}\sigma^\rho - \sigma^\rho\bar\sigma^{\mu\nu} = \eta^{\mu\rho}\sigma^\nu - \eta^{\nu\rho}\sigma^\mu.$$

   (g) [1 mark] Explain what is a superfield.

   (h) [5 marks] Write down the most general massless multiplet of $\mathcal{N} = 4$ rigid supersymmetry, with the particles organised in representations of the $SU(4)_R$ R-symmetry. Find also the most general massless multiplet of $\mathcal{N} = 3$ rigid supersymmetry, and organise the particles in representations of the $U(3)_R \cong SU(3)_R \times U(1)_R$ R-symmetry. Work out how the $\mathcal{N} = 3$ multiplet is embedded into the $\mathcal{N} = 4$ multiplet.

   [*The following mathematical fact may be useful: the branching rules for the Lie algebra decomposition* $SU(4) \to SU(3) \times U(1)$ *are:*

$$\mathbf{4} \to \mathbf{1}_{-3} \oplus \mathbf{3}_1 \ , \qquad \mathbf{6} \to \mathbf{3}_{-2} \oplus \bar{\mathbf{3}}_2 \ , \qquad \mathbf{10} \to \mathbf{1}_{-6} \oplus \mathbf{3}_{-2} \oplus \mathbf{6}_2 \ ,$$

   for the first few irreducible representations of $SU(4)$.]

2. (a) [3 marks] Give the definition of the chiral superfield, Φ, as one satisfying a differential constraint in superspace. Define the relevant differential operators you use. Then, write down Φ in components schematically (that is, up to numerical coefficients).

   (b) [3 marks] Give the definition of the vector superfield, V, and write it down in components in the Wess-Zumino (WZ) gauge, schematically.

   (c) [1 mark] Compute V^3 (the third power of V) in the WZ gauge.

   (d) [5 marks] Write down the Lagrangian of 4d $\mathcal{N} = 1$ massless SQCD ($SU(N_c)$ with N_f flavors) in superspace, and explain what types of interactions are contained in each term. You may use the notation:

$$\mathcal{W}_\alpha = \frac{i}{8}\bar{\mathrm{D}}\bar{\mathrm{D}}e^{-2V}\mathrm{D}_\alpha e^{2V} \ . \tag{2}$$

Then, assuming $N_c > 3$, write down the most general renormalisable superpotential one could add to deform SQCD in the UV.

(e) [3 marks] Give the global symmetries of the massless SQCD Lagrangian, for $N_c > 2$. Which symmetries survive quantum mechanically, and why? Explain why, in the quantum theory, one can determine the R-charge of the "matter" chiral multiplets to be:

$$r = 1 - \frac{N_c}{N_f} \ .$$

(f) [3 marks] Consider the theory of a single chiral multiplet, M, with superpotential:

$$W = \frac{\Lambda^5}{M} + mM \ . \tag{3}$$

If the coupling constants Λ and m have engineering dimension 1, what is the engineering dimension of M? Analyse the vacuum structure of this theory. (For simplicity, you may assume the Kähler potential is canonical.) Explain what happens in the limit $m \to 0$.

(g) [7 marks] Consider the theory of 15 chiral multiplets $M_{ij} = -M_{ji}$, which transform into the two-index anti-symmetric (**15**) representation of an $SU(6)$ flavor symmetry, with the following canonical Kähler potential and cubic superpotential:

$$K = \frac{1}{2}\bar{M}^{ij} M_{ij} \ , \qquad W = \frac{g_0}{48}\epsilon^{ijklmn} M_{ij} M_{kl} M_{mn} \ , \tag{4}$$

where ϵ^{ijklmn} is the fully anti-symmetric symbol (an invariant tensor of $SU(6)$). The indices $i, j, \cdots$ runs over $i = 1, \cdots, 6$. Assume $g_0 \in \mathbb{R}$.

- Write down the complete Lagrangian that follows from this superpotential *in components,* for the 15 scalar fields M_{ij}, fermions ψ_{ij}, and auxiliary fields F_{ij}.
- Integrate out the auxiliary fields and write down the resulting interaction Lagrangian (for simplicity, you may write the answer up to numerical factors).
- This theory provides an infrared (IR) description of SQCD with gauge group $SU(2)$ and $N_f = 3$. Give at least two detailed symmetry arguments in favor of this identification. [*Hint: the fundamental and anti-fundamental representations of* $SU(2)$ *are isomorphic.*]
- Consider adding the following linear superpotential term to the theory:

$$W_{\text{lin}} = mM_{56} + \mu \left(\sum_{a=1}^{4} M_{a5} + \sum_{a=1}^{4} M_{a6} + \sum_{a=1}^{4} \sum_{b=a+1}^{4} M_{ab} \right) \ , \tag{5}$$

with indices $a, b = 1, \cdots, 4$. We claim that, by integrating out the 14 fields M_{ij} with $(ij) \neq (56)$ from the superpotential, in the limit $\mu \gg m$, one finds the theory of a single chiral multiplet $M \equiv M_{56}$ given in equation (3). Give a *schematic* derivation of this fact, assuming a supersymmetric vacuum exists. How does it relate to the IR physics of SQCD?