Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

QUANTUM MATTER

Trinity Term 2022

Wednesday, 20th April 2022, 9:30am-11:30am

You should submit answers to both questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

You are permitted to use the following material(s):
One summary sheet of A4 notes

The use of a calculator is **not** *allowed.*

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. Consider a system of N spinless bosons in a box with periodic boundary conditions and size $L \times L \times L$. The Hamiltonian is given by a kinetic term plus an interaction term

$$H = H_{kin} + H_{int} = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} \quad + \quad U \sum_{1 \leqslant i < j \leqslant N} \nabla^2 \delta(\mathbf{r}_i - \mathbf{r}_j) \qquad ,$$

where $\delta()$ is a three-dimensional Dirac delta function and U is a constant.

(a) Write this Hamiltonian in second-quantized notation using bosonic creation and annihilation operators $\hat{\psi}^\dagger(\mathbf{r})$ and $\hat{\psi}(\mathbf{r})$. Give the commutation relations of these operators. Then rewrite the Hamiltonian in second-quantized notation using bosonic plane-wave creation and annihilation operators $c_{\mathbf{k}}^\dagger$ and $c_{\mathbf{k}}$ where $\mathbf{k}$ indicates a wavevector. Give the commutation relations of these operators and also indicate what values are allowed for the wavevectors $\mathbf{k}$. (Hint: handle ∇^2 with integration by parts.) [8]

(b) Consider a state of this system defined in an occupation basis. We write

$$|\Psi\rangle = |n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, n_{\mathbf{k}_3}, \ldots\rangle \qquad \text{such that} \qquad \sum_{\mathbf{k}} n_{\mathbf{k}} = N \qquad , \tag{1}$$

where $n_{\mathbf{k}}$ is a nonnegative integer giving the occupancy of plane-wave state $\mathbf{k}$. Calculate the expectation value of the kinetic and interaction energies

$$E_{kin} = \frac{\langle\Psi|H_{kin}|\Psi\rangle}{\langle\Psi|\Psi\rangle} \qquad \text{and} \qquad E_{int} = \frac{\langle\Psi|H_{int}|\Psi\rangle}{\langle\Psi|\Psi\rangle} \qquad .$$

[8]

(c) Rewrite your result as

$$E_{int} = \lambda E_{kin} + X \qquad ,$$

where λ is a constant (independent of the occupancies $n_{\mathbf{k}}$) and X is a function of the occupancies but X vanishes if the total wavevector $\sum_{\mathbf{k}} \mathbf{k} n_{\mathbf{k}}$ vanishes. Determine λ and X. [3]

(d) Let us restrict our attention to cases where the total wavevector does vanish. We can then write $E_{kin} + E_{int} = (1 + \lambda) E_{kin}$ for all possible occupancies in Eq. (1). Explain either why it is accurate, or why it is not accurate, to model the system as noninteracting bosons of mass

$$\frac{1}{2m^*} = (1 + \lambda) \frac{1}{2m} \qquad .$$

[3]

(e) Explaining your reasoning, state whether this system should be superfluid at low temperature. This does not need to be a complete calculation, but should be a convincing argument. [3]

2. Consider a system of electrons of mass m and charge $-e$, where the electrons are confined to move in a two-dimensional plane. The two-dimensional density of electrons is ρ and we assume the system has no disorder.

In parts (a) and (b) we should assume that the electrons do not interact with each other via the Coulomb interaction, although they do respond to an externally applied electric field.

(a) Using Drude theory, or otherwise, calculate the conductivity of this system at finite frequency ω and zero wavevector (long wavelength). [5]

(b) The density-density response function $\chi(\mathbf{q}, \omega)$ at finite wavevector $\mathbf{q}$ and frequency ω is defined as

$$\delta n(\mathbf{q}, \omega) = \chi(\mathbf{q}, \omega)\, \phi_{ext}(\mathbf{q}, \omega) \quad ,$$

where ϕ_{ext} is an externally applied electrostatic potential and δn is the resulting induced charge density (both of these being defined at wavevector $\mathbf{q}$ and frequency ω). Still assuming that the electrons do not interact with each other, using the result of part (a), or otherwise, calculate $\chi(\mathbf{q}, \omega)$ in the limit of small wavevector. [6]

(c) Now let us allow the electrons to interact with each other via the usual Coulomb interaction $V(\mathbf{r}) = e^2/(4\pi\epsilon|\mathbf{r}|)$. Using the Random Phase Approximation (RPA) or otherwise, determine the density-density response function for the interacting electrons. [6]

(d) Determine the dispersion $\omega(\mathbf{q})$ of the plasmon excitation. Determine the group velocity of plasmons as a function of $|\mathbf{q}|$. What problem occurs at very small $|\mathbf{q}|$? Explain why our calculation becomes incorrect. [8]

The following information may be helpful:

$$\int d^2r\; e^{i\mathbf{q}\cdot\mathbf{r}} \frac{1}{|\mathbf{r}|} = \frac{2\pi}{|\mathbf{q}|} \quad .$$