A15758W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

QUANTUM MATTER: SUPERCONDUCTORS, SUPERFLUIDS, AND FERMI LIQUIDS

Trinity Term 2019

TUESDAY, 11TH JUNE 2019, 2:30pm to 4:30pm

You should submit answers to both questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. (a) [4 marks] Give a definition of Bose condensation, and explain how it differs from superfluidity. (Answers should be a short paragraph or less).

   (b) [4 marks] Consider a gas of N noninteracting spinless bosons moving in d dimensions with the dispersion relation

$$\epsilon(\mathbf{k}) = \alpha|\mathbf{k}|^4 ,$$

   where ϵ is the energy and $\mathbf{k}$ is the wavevector. For what value of the dimension d do you expect the system to exhibit Bose condensation at sufficiently low (but nonzero) temperature?

   (c) [4 marks] Now assume instead that the N bosons in the system interact with each other via a short-range repulsive interaction, so that the Hamiltonian is given by

$$H = \sum_{i=1}^{N} \alpha|\mathbf{k}_i|^4 + U \sum_{i<j}^{N} \delta(\mathbf{r}_i - \mathbf{r}_j)$$

   with $U > 0$. Rewrite this Hamiltonian in second quantized form using bosonic operators $\hat{\psi}^\dagger(\mathbf{r})$ and $\hat{\psi}(\mathbf{r})$ satisfying commutation relations

$$[\hat{\psi}(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{r}')] = \delta(\mathbf{r} - \mathbf{r}') .$$

   (d) [4 marks] Using the equations of motion

$$i\hbar \frac{\partial \hat{\psi}(\mathbf{r})}{\partial t} = [\hat{\psi}(\mathbf{r}), H] ,$$

   derive a (time-dependent) Gross-Pitaevskii equation for this system. The Gross-Pitaevskii equation should be written in terms of a classical field $\psi(\mathbf{r})$ rather than an operator $\hat{\psi}(\mathbf{r})$. The classical field ψ should be normalized so that $|\psi(\mathbf{r})|^2$ is the boson density at position $\mathbf{r}$. Make sure to explain how one gets to a classical field from the quantum operator.

   (e) [5 marks] The ground state is given by $|\psi(\mathbf{r})| = \psi_0$ a constant. Write a low-energy excitation at wavevector $\mathbf{k}$ in the form

$$\psi(\mathbf{r}, t) = e^{-i\mu t} \left[\psi_0 + \psi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} + \psi_{-\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{r} + i\omega t} \right] .$$

   Assuming that $\psi_{\mathbf{k}}$ and $\psi_{-\mathbf{k}}$ are small, substitute this form into the Gross-Pitaevskii equation to calculate the low-energy excitation spectrum (i.e., the dispersion relation) of this interacting Bose gas.

   (f) [4 marks] Given the dispersion relation that you just derived, do you expect this system will be a superfluid? Justify your answer.

2. (a) [4 marks] Explain what is meant by Hartree-Fock approximation.

(b) [5 marks] Consider a Hamiltonian of the following general form

$$H = \sum_{i,j} h_{ij} c_i^\dagger c_j + \frac{1}{2} \sum_{ijkl} v_{ijkl} \; c_i^\dagger c_j^\dagger c_l c_k \; ,$$

where $c_i^\dagger$ and c_j are fermion creation and annihilation operators satisfying

$$\{c_i, c_j^\dagger\} = \delta_{ij}$$

with indices $i, j, \ldots$ representing arbitrary orbitals. For a fixed number N of electrons in the system, give a set of self-consistent equations whose solution would give the Hartree-Fock ground state of the system (i.e, explain how in principle one finds the Hartree-Fock ground state).

(c) [7 marks] Consider now the two-site Hubbard model for spinful fermions:

$$H = -t \sum_\sigma (c_{1\sigma}^\dagger c_{2\sigma} + c_{2\sigma}^\dagger c_{1\sigma}) + U \sum_i (c_{i\uparrow}^\dagger c_{i\uparrow})(c_{i\downarrow}^\dagger c_{i\downarrow}) \; .$$

Without loss of generality you may assume t is real and $t \geqslant 0$. Consider the case of $N = 2$ fermions. By explicitly constructing the ground-state wavefunction,

(1) show that for $U = 0$ (with $t \neq 0$), the Hartree-Fock approximation gives the exact ground-state energy;

(2) show that for $t = 0$, the Hartree-Fock approximation gives the exact ground-state energy for both $U > 0$ and $U < 0$.

What is the ground-state degeneracy in each of the cases you just considered?

(d) [9 marks] Again consider the case with $N = 2$ fermions. For the case where $U \neq 0$ and $t \neq 0$, the ground state is never degenerate. You can take this statement as given. Give an argument that the exact ground state in this case must be spin zero. Argue that any Hartree-Fock ground state with zero spin must be expressable in the form $f_1^\dagger f_2^\dagger |0\rangle$ where

$$\begin{aligned} f_1^\dagger &= \alpha c_{1\uparrow}^\dagger + \beta c_{2\uparrow}^\dagger \; , \\ f_2^\dagger &= \alpha' c_{1\downarrow}^\dagger + \beta' c_{2\downarrow}^\dagger \; . \end{aligned}$$

Considering the symmetries of the Hamiltonian, what values can $\alpha, \alpha', \beta, \beta'$ take for a Hartree-Fock ground state? For $U \neq 0$ and $t \neq 0$, is the Hartree-Fock approximation ever exact? Prove your answer.