Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

QUANTUM MATTER: SUPERCONDUCTORS, SUPERFLUIDS, AND FERMI LIQUIDS

Trinity Term 2018

MONDAY, 4TH JUNE 2018, 2:30pm to 4:30pm

You should submit answers to both questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. In this problem we consider a system of spinless bosons in D dimensions with creation and annihilation operators satisfying canonical commutations

$$[\hat{\psi}(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{r}')] = \delta(\mathbf{r} - \mathbf{r}')$$

where here δ is a D-dimensional delta function.

Consider a complete basis of orthonormal single particle orbitals $\phi_n(\mathbf{r})$. Define a set of creation operators in terms of these orbitals and the above operators $\hat{\psi}^\dagger$ given by

$$a_n^\dagger = \int \mathbf{dr}\, \phi_n(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r})$$

(a) [2 marks] Show that the operator $a_n^\dagger$ satisfies the commutations $[a_n, a_m^\dagger] = \delta_{nm}$.

(b) [3 marks] Derive an equation giving $\psi^\dagger(\mathbf{r})$ in terms of the operators $a_m^\dagger$. (Hint: You will need to use completeness of the basis ϕ_n.) Write a general expression for the boson density operator $\rho(\mathbf{r})$ in terms of the a and $a^\dagger$ operators.

Consider the coherent state

$$|\alpha, n\rangle = e^{\alpha a_n^\dagger}|0\rangle$$

where α is a complex constant.

(c) [2 marks] By using the Taylor expansion of the exponential or otherwise, show that

$$a_m|\alpha, n\rangle = \alpha\, \delta_{nm}|\alpha, n\rangle$$

and determine the expectation of the density $\rho(\mathbf{r})$ described by the coherent state.

Now consider a Hamiltonian for the system of bosons given by

$$H = \sum_{1 \leqslant i \leqslant N} \left[\frac{\mathbf{p}_i^2}{2m} - \epsilon_0\right] + V_0 \sum_{1 \leqslant i < j \leqslant N} \delta(\mathbf{r}_i - \mathbf{r}_j)$$

where V_0 is the interparticle interaction which we will assume is repulsive and δ is the D-dimensional delta function.

(d) [4 marks] Rewrite the Hamiltonian H in second quantized form using creation and annihilation operators $\hat{\psi}^\dagger(\mathbf{r})$ and $\hat{\psi}(\mathbf{r})$.

(e) [7 marks] By taking the expectation of the above Hamiltonian with respect to the above coherent state, or otherwise, derive the form of the zero temperature Ginzburg-Landau (free) energy

$$F = \int \mathbf{dr} \left\{\kappa|\nabla\psi(\mathbf{r})|^2 + \alpha_2|\psi(\mathbf{r})|^2 + \alpha_4|\psi(\mathbf{r})|^4\right\}$$

where here $\psi(\mathbf{r})$ is a complex scalar. How is this scalar related to the above second quantized operators? Determine the values of the coefficients κ, α_2, and α_4.

(f) [2 marks] Determine the value of $|\psi|$ in the ground state if the system is translationally invariant (i.e., you may assume periodic boundary conditions). Call this value ψ_0. Determine the free energy per unit volume of the system in the ground state. You may assume that $\epsilon_0 < 0$.

(g) [3 marks] Using functional differentiation, or otherwise, derive from the free energy a non-linear time-independent Schrödinger equation (otherwise known as a Gross-Pitaevskii equation).

(h) [2 marks] Define a new field $f = \psi/\psi_0$ and rewrite the above-derived non-linear Schrödinger equation in terms of this new field. Using this equation or otherwise, determine the characteristic length scale ξ, known as the coherence length.

2. Consider a one dimensional gas of interacting spin 1/2 fermions with Hamiltonian

$$H = \sum_{1 \leqslant i \leqslant N} \frac{p_i^2}{2m} + \sum_{1 \leqslant i < j \leqslant N} V(x_i - x_j).$$

Assume periodic boundary conditions in a system of length L which is very large. The fermions have total number density $n = N/L$, and throughout this question you may assume zero temperature. Let us also define

$$\tilde{V}_q = \int dx \, e^{iqx} \, V(x).$$

(a) [3 marks] Rewrite the Hamiltonian in second quantized notation using creation and annihilation operators which have anticommutation relations $\{\hat{\psi}_\sigma(x), \hat{\psi}^\dagger_{\sigma'}(x')\} = \delta_{\sigma,\sigma'}\delta(x - x')$ where σ is the spin index taking the values $\uparrow$ and $\downarrow$.

(b) [3 marks] Rewrite the Hamiltonian in second quantized notation using k-space creation and annihilation operators having anticommutation relations $\{c_{k,\sigma}, c^\dagger_{q,\sigma'}\} = \delta_{\sigma,\sigma'}\delta_{kq}$ where the wavevector is quantized as $k = 2\pi n/L$ with integer n.

(c) [3 marks] Give a definition of Hartree-Fock approximation, and explain when it is identical to first order perturbation theory.

(d) [8 marks] To first order in perturbation theory, or in Hartree-Fock, the effective Hamiltonian can be written as

$$H = \sum_{k,\sigma} \epsilon(k) \, c^\dagger_{k,\sigma} c_{k,\sigma}$$

where

$$\epsilon(k) = \frac{\hbar^2 k^2}{2m} + X + \Sigma(k)$$

Give expressions for X and $\Sigma(k)$ in terms of $\tilde{V}_q$ and $n(k)$ the expectation of the occupancy of state k.

(e) [2 marks] The effective mass can be written as

$$\frac{\hbar^2}{m^*} = \frac{\hbar^2}{m} + \frac{\partial^2 \Sigma(k)}{\partial k^2}.$$

Write an expression for this effective mass in terms of $\tilde{V}_q$ that does not involve an integration over k (or sum with infinite number of terms in the thermodynamic limit). Hint: Cancel the integration with a derivative.

(f) [2 marks] For $V(x) \sim 1/\sqrt{|x|}$ the Fourier transform is $\tilde{V}_q = A/\sqrt{|q|}$ with some constant A. Determine the effective mass at the Fermi surface.

(g) [4 marks] Consider an interacting one-dimensional Fermi sea at rest, with the interaction as stated above in part (f). In the spirit of the arguments used in Landau Fermi liquid theory, determine the total energy of moving a small density n_0 of fermions from the Fermi surface at $-k_F$ to the Fermi surface at $+k_F$. Assume that you move an equal number of spin up and spin down fermions.