A15088W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

QUANTUM FIELD THEORY

Hilary Term 2022

WEDNESDAY, 12th JANUARY 2022, 09:30 am to 12:30 pm

You should submit answers to all three questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. The Lagrangian density for a real scalar field theory *in 6 dimensions* (five space plus time) is given by

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{g_1}{3!}\phi^3 - \frac{g_2}{4!}\phi^4.$$

The superficial degree of divergence of a graph with n external lines, n_3 three-point vertices, and n_4 four-point vertices is given by

$$\omega = 6 - n d_\phi - d_{g_1} n_3 - d_{g_2} n_4.$$

(a) [3 marks] What are the (length) dimensions d_ϕ, d_{g_1}, d_{g_2} of ϕ, g_1 and g_2 respectively?

(b) [3 marks] Write down the Feynman rules in momentum space.

(c) [6 marks] By analysing the possible one-particle irreducible divergent graphs explain why the theory is expected to be renormalizable if $g_2 = 0$. Write down the Lagrangian density including counter terms.

(d) [13 marks] Find the divergent parts of the one-loop counter terms in a momentum cut-off regularization.

[You may assume the identities :

$$\int^\Lambda \frac{1}{(p^2 - K + i\epsilon)^{1+n}} \frac{d^6p}{(2\pi)^6} = -i\frac{C}{n!}\frac{d^n}{dK^n}\left(\frac{1}{2}\Lambda^4 - K\Lambda^2 + K^2\log\frac{\Lambda^2}{K}\right),$$

where C is a constant; and

$$\frac{1}{(p-k)^2 - m^2}\frac{1}{(p+k)^2 - m^2} = \frac{1}{(p^2 - m^2)^2}\left(1 - \frac{1}{2}\frac{k^2}{(p^2 - m^2)} - \frac{(k.p)^2}{(p^2 - m^2)^2} + O(k^4)\right).]$$

2. The Lagrangian density for a system in four space-time dimensions consisting of a scalar field σ, whose action has a mass term but not a kinetic term, and a Dirac fermion ψ is given by

$$\mathcal{L} = -\frac{1}{2}\sigma^2 + \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi + g\sigma\overline{\psi}\psi.$$

(a) [4 marks] Write down the momentum space Feynman rules assuming Minkowski space-time.

(b) [4 marks] Draw the tree-level Feynman graphs, including momentum labelling, for the following processes:
i) scattering of two fermions;
ii) scattering of a fermion and an anti-fermion.

(c) [13 marks] A fermion with spin and four-momentum s_1, p_1 scatters off an anti-fermion with spin and four-momentum s_2, p_2 producing a fermion with spin and four-momentum s_3, p_3 and an anti-fermion with spin and four-momentum s_4, p_4 . Write down a formula for the matrix element $M^{s_1,s_2,s_3,s_4}_{p_1,p_2,p_3,p_4}$. The total amplitude squared for the scattering of unpolarised particles to a final state of any spin is given by

$$|M|^2 = \frac{1}{4}\sum_{s_1,s_2,s_3,s_4} |M^{s_1,s_2,s_3,s_4}_{p_1,p_2,p_3,p_4}|^2.$$

Show that

$$|M|^2 = \frac{1}{2}g^4\left(3(t-4m^2)^2 + 3(s-4m^2)^2 - (u-2m^2)^2 - 4m^4\right),$$

where $s = (p_1+p_2)^2$, $t = (p_1-p_3)^2$, $u = (p_1-p_4)^2$. Explain why this quantity is symmetric under exchange of s and t. [You may assume that: $\sum_s u^s_a(p)\overline{u}^s_b(p) = (\not{p}+m)_{ab}$ and that $\sum_s v^s_a(p)\overline{v}^s_b(p) = (\not{p}-m)_{ab}$; and the trace identities $\mathrm{Tr}\gamma^\mu\gamma^\nu = 4g^{\mu\nu}$, $\mathrm{Tr}\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho = 4(g^{\mu\nu}g^{\lambda\rho} - g^{\mu\lambda}g^{\nu\rho} + g^{\mu\rho}g^{\nu\lambda})$]

(d) [4 marks] Show that the Green's function with n external fermion lines, and no external σ lines, is the same as the corresponding Green's function computed with the Lagrangian density

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi + \frac{1}{2}g^2(\overline{\psi}\psi)^2.$$

3. The Hamiltonian for Dirac fermions is given by

$$H = \int d^3\mathbf{x}\, \overline{\psi}(-i\gamma^k \partial_k + m)\psi,$$

where k is a space index, and at fixed time

$$\{\psi_a^\dagger(\mathbf{x}'), \psi_b(\mathbf{x})\} = \delta_{ab}\, \delta^3(\mathbf{x}' - \mathbf{x}),$$

where a, b are spinor indices.

(a) [6 marks] Show that

$$[X, YZ] = \{X, Y\}Z - Y\{X, Z\},$$

and hence that

$$[VX, YZ] = V\{X, Y\}Z - VY\{X, Z\} + \{V, Y\}ZX - Y\{V, Z\}X\,,$$

where V, X, Y, Z are operators. Compute

$$[\psi_a^\dagger(\mathbf{x})\psi_a(\mathbf{x}), \psi_b^\dagger(\mathbf{y})\psi_c(\mathbf{z})].$$

(b) [9 marks] Compute the commutator $C = [\psi^\dagger\psi, H]$. Hence, and *not* otherwise, show that the operator C is the divergence of a current operator $\mathbf{J}$ and find an expression for $\mathbf{J}$.

(c) [4 marks] Explain your result in part b) in the context of the symmetries of H.

(d) [6 marks] How many similar conserved currents are there for a system of two Dirac fermions with Hamiltonian

$$H = \int d^3\mathbf{x} \sum_{A=1}^{2} \overline{\psi}^A(-i\gamma^k \partial_k + m)\psi^A \quad ?$$