# Examiners' Report: Final Honour School of Mathematical and Theoretical Physics Part C and MSc in Mathematical and Theoretical Physics Trinity Term 2021

November 8, 2021

# Part I

## A. STATISTICS

- **Numbers and percentages in each class.**

  See Table 1.

| | Numbers | | | | | | Percentages % | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 2021 | 2020 | (2019) | (2018) | (2017) | (2016) | 2021 | 2020 | (2019) | (2018) | (2017) | (2016) |
| Distinction | 42 | (42) | (40) | (25) | (31) | (18) | 63 | (76) | (76) | (60) | (76) | (86) |
| Merit | 10 | (9) | (6) | (n/a) | (n/a) | (n/a) | 15 | (17) | (11) | (n/a) | (n/a) | (n/a) |
| Pass | 12 | (3) | (6) | (17) | (10) | (3) | 18 | (5) | (11) | (40) | (24) | (14) |
| Fail | 3 | (1) | (1) | (1) | (0) | (0) | 4 | (2) | (0) | (0) | (0) | (0) |
| Total | 67 | (55) | (53) | (42) | (41) | (21) | 100 | (100) | (100) | (100) | (100) | (100) |

Table 1: Numbers and percentages in each class

- **Numbers of vivas and effects of vivas on classes of result.**
  No vivas were held.

- **Marking of scripts.**
  All dissertations and three mini-project subjects were double-marked, after which the two markers consulted in order to agree a mark between them.

  All written examinations and take-home exams were single-marked according to carefully checked model solutions and a pre-defined marking scheme which was closely adhered to. One mini project subject which followed a mark scheme (Galactic and Planetary Dynamics) was also marked in the same way. A comprehensive independent checking procedure is also followed.

## B. New examining methods and procedures

Due to the pandemic, new procedures were introduced. Written examinations in Trinity term took place in the form of timed open-book examinations, where students had the same length of time to complete the open-book examination as they would have had for a written examination, plus 30 minutes to upload/download the examination paper, and to scan and submit their solutions. Students were required to uphold an honour code.

## C. Changes in examining methods and procedures currently under discussion or contemplated for the future

None.

## D. Notice of examination conventions for candidates

Notices to candidates were sent on: 2nd October 2020 (first notice), 24th November 2020 (second notice),1st March 2021 (third notice), 5th May 2021 (fourth notice) and 17th May 2021 (final notice).

The examination conventions for 2020-2021 are on-line at http://mmathphys.physics.ox.ac.uk/students.

# Part II

## A. General Comments on the Examination

## B. Equality and Diversity issues and breakdown of the results by gender

Removed from public version.

## C. Detailed numbers on candidates' performance in each part of the examination

The number of candidates taking each paper is shown in Table 2 and in the Average USM per Formal Assessment graph below. In accordance with University guidelines, statistics are not given for papers where the number of candidates was five or fewer.

Table 2: Numbers taking each paper

| Paper | Number of Candidates | Avg USM | StDev USM |
|---|---|---|---|
| Advanced Fluid Dynamics | - | - | - |
| Advanced Philosophy of Physics | - | - | - |
| Advanced Quantum Field Theory | 41 | 70 | 13.5 |
| Advanced Quantum Theory | 27 | 68 | 17.6 |
| Algebraic Geometry | - | - | - |
| Algebraic Topology | - | - | - |
| Applied Complex Variables | - | - | - |
| Approximation of Functions | - | - | - |
| Collisionless Plasma Physics | | | |
| Computational Algebraic Topology | - | - | - |
| Continuous Optimisation | - | - | - |
| Differentiable Manifolds | 12 | 72 | 29 |
| Dissertation (single unit) | 22 | 76 | 9 |
| Dissertation (double unit) | 21 | 78 | 8 |
| Elliptic Curves | - | - | |
| Homological Algebra | - | - | - |
| Galactic and Planetary Dynamics | | | |
| General Relativity I | 33 | 63 | 23 |
| General Relativity II | 19 | 67 | 12 |
| Geophysical Fluid Dynamics | - | - | |
| Geometric Group Theory | - | - | - |
| Groups and Representations | 52 | 71 | 13 |
| Introduction to Quantum Information | 29 | 63 | 22 |
| Introduction to Schemes | - | - | - |
| Kinetic Theory | - | - | - |
| Networks | 9 | 73 | 7.4 |
| Numerical Linear Algebra | - | - | - |
| Perturbation Methods | 8 | 66 | 15 |
| Quantum Field Theory | 67 | 71 | 9.6 |
| Quantum Matter | 17 | 67 | 13 |
| Radiative Processes and High Eng. Astro. | - | - | - |
| Random Matrix Theory | 8 | 76 | 15 |
| Stochastic Differential Equations | - | - | - |
| String Theory I | 29 | 71 | 5 |
| Supersymmetry and Supergravity | 7 | 73 | 29.4 |
| Theories of Deep Learning | - | - | - |

The number of candidates taking each homework completion course is shown in Table 3. In accordance with University guidelines, statistics are not given for papers where the number of candidates was five or fewer.

Table 3: Numbers taking each homework completion course

| Paper | Number of Candidates | Percentage completing course |
|---|---|---|
| Advanced Fluid Dynamics | - | - |
| Advanced Quantum Theory | - | - |
| Astroparticle Physics | 9 | 100 |
| Collisionless Plasma Physics | - | - |
| Conformal Field Theory | 31 | 100 |
| Cosmology | 20 | 100 |
| Galactic and Planetary Dynamics | - | - |
| Group and Representations | 51 | 100 |
| High Energy Density Physics | - | - |
| Kinetic Theory | - | - |
| Nonequilibrium Statistical Physics | 9 | 100 |
| Quantum Field Theory in Curved Space Time | 22 | 100 |
| Quantum Matter | - | - |
| Renormalisation Group | 17 | 94 |
| Soft Matter Physics | | 100 |
| String Theory II | 8 | 100 |
| Supersymmetry and Supergravity | 14 | 100 |
| Symbolic, Numerical and Graphical Scientific Programming | 7 | 100 |
| The Standard Model and Beyond I | 11 | 91 |
| The Standard Model and Beyond II | - | 100 |
| Topological Quantum Theory | 23 | 100 |

## D. Assessors' comments on sections and on individual questions

### Advanced Fluid Dynamics

Q1 Part (a). All students knew basically what to do, but most lost quite a bit of time because they chose to copy far more than needed from the lecture notes, despite being urged by the script not to do so. There was some low-level confusion about perturbing and differentiating **b**. Some lost marks because the part of the question asking them to show that the reduced equations for the Alfvénic fields were unaffected by the introduction of anisotropic pressure was ignored.

Part (b). Everybody understood how to linearise the CGL equations. Nobody seemed to realise that the 5th equation in the set was the perpendicular pressure balance, connecting $\delta p_\perp$ to $\delta B$. Consequently, no one derived the right dispersion relation.

Part (c). Very few students attempted this part and saw the basic physical point although perhaps a little vaguely (the restoring force is $-\nabla \delta p_\parallel$; in the CGL scheme, $\delta p_\parallel$ is not hard-coupled to $\delta p_\perp$ and, thereby, to $\delta B$ via the perpendicular pressure balance, so it can be a lot larger than in the isotropic-pressure case; the "slow" wave can consequently propagate at $\sim$ sound speed).

Q2 All parts received at least one completely successful attempt, though no candidate completed all parts successfully.

Part (a) was mostly done well, though a few candidates includes time derivatives and did not justify why these later vanished. The expected interpretation is that the viscous dissipation is instantaneously equal to the rate of working by surface stresses on the boundary.

Part (b) was also mostly done well, though a few candidates incorrectly relied upon a nonexistent "linearity" property for the dissipations due to two different flows. Instead, one should write down the dissipation integral from part (a) for the volume of fluid outside the particles, then use $\mathbf{u} = \mathbf{u}^{(0)}$ on the outer boundary $S_0$.

In part (c) several candidates incorrectly tried to apply the reciprocal theorem to individual surfaces, the outer boundary $S_0$ and a particle surface $S_p$, rather than to the complete bounding surface of a fluid volume. Several candidates omitted the second part, to show that the contribution from $\mathbf{u} \cdot \sigma^{(0)} \cdot \mathbf{n}$ vanishes for rigid particles. The intended approach relies upon $\mathbf{u} = \mathbf{U}_p + \mathbf{\Omega}_p \times (\mathbf{x} - \mathbf{x}_p)$ being a rigid-body motion on the boundary $S_p$ to simplify the surface integral. All successful attempts from candidates relied instead on the divergence theorem, and implicitly defined $\mathbf{u}$ to be a rigid-body motion inside the particle (where there is no fluid) so that the strain $\mathbf{e} = 0$ inside the particle. Full marks were given for this approach.

In part (d) several candidates asserted that the contributions from $\mathbf{u}_p^{(0)}$ and $\mathbf{\Omega}_p^{(0)}$ vanish identically due to vector calculus identities. Instead, they vanish because the particles are force-free and torque-free.

Part (e) was mostly done well by those who attempted it, though a few candidates appeared to have relied upon the end result to evaluate the surface integral and/or the volume fraction occupied by particles in intermediate steps.

**Advanced Quantum Field Theory**

Question 1 related to 1–loop calculations within scalar QED. Parts (ii) and (iv) were answered reasonably well. The principle issues here came from algebra mistakes, and/or time constraints which could lead to the student not managing to get to the final answers. Parts (i) and (iii) are rather straightforward, and were answered very well in general. Part (v) was answered well in general, although with some students forgetting to answer the part relating to $Z_2$ and/or not quite understanding the conceptual issue relating to the $\xi$ dependence of observables.

Question 2 (a) related to a tree-level scattering calculation. Part (i) was rather straightforward and was answered very well in general. Part (ii) was from a conceptual point of view an application of techniques covered in detail in the course. However, the fact that both initial and final state particles were kept as massive lead to the algebra being a little too intricate for many students, although a large fraction of marks were available (and gained) for carrying out the majority of the calculation, even if the student struggled to arrive at the exact results due to an algebra issue relating to this. In a minority of cases students struggled to set the problem up, perhaps relating to the mixing of quark and QED interactions. Part (iii) required an essentially textbook application of colour algebra, and was answered reasonably well. Part (iv) was more challenging, as such a combination of both QED and QCD diagrams (and their lack of interference) was not explicitly covered in the course. Although some students spotted this and did well, many struggled with this part, or did not answer.

Question 2 (b) related to a different tree–level calculation in the t'Hooft Veltman gauge. Almost all students could set the problem up, and achieve some marks, and a reasonable number achieved full marks.

Question 3 (a) related to spontaneous symmetry breaking. In some cases this was answered well, but not by many students. This was in many cases clearly related to a lack of time available to answer, i.e. in some cases almost nothing was written. However in part it may have been related to the fact that this was covered quite late on in the course and was a somewhat challenging question.

Question 3 (b) related to manipulations of fermionic path integrals. This was well covered in the course (and indeed in general terms in the QFT course) and was very well answered in the majority of cases.

**Advanced Quantum Theory**

A question on obtaining the path integral for the simple harmonic oscillator. Most students gave excellent answers to parts (a-c). In part (b), since generous hints were given in the question, details of the derivation were important to receive full marks; several students were penalized for not justifying why linear time derivative terms vanish (due to the equations of motion and the boundary conditions) or explaining why the change in integration measure is trivial. When deriving the classical action in part (c), marks were subtracted for answers with variations on the theme of "and after some algebra we see that..." since the final answer was stated in the question. A handful of students gave an elegant solution involving evaluating $x\dot{x}|_0^t$, which received full marks as long as it was properly explained. Fewer students gave a satisfactory answer to part (d). The key idea, as suggested by the hint, is to write the Schrödinger equation for the unitary time evolution, and then insert sufficient resolutions of the identity in position space to write this as an integral equation whose eigenfunction is the ground state, and use the path integral to simplify it. Many students started correctly but then got confused over how to complete the calculation, with one or two missing the fact that the necessary Gaussian integral was provided in the question. Finally, part (e) was attempted by many students but in spme cases they simply quoted the free particle result without explaining how it arises as a limiting case of the harmonic oscillator, which only received partial marks. The distribution of marks was roughly consistent with that envisaged.

A question on Lifshitz transitions within Landau theory. Nearly all students gave a satisfactory answer to part (a). Part (b) was the meat of the question, and should have been a straightforward step-by-step application of Landau theory. The simplest route to establishing the phase boundaries (i.e., computing free energies) and the fact that there are only 3 phases were both spelled out in the problem. However, most students seemed to struggle with this, often finding 4 phases by not recognizing that the equation setting $k_0$ is only meaningful in an ordered phases, since otherwise the saddle point equations are automatically satisfied. Students who did compute the free energies for the most part mapped the phase diagram correctly, though a significant minority made an error at this stage. (As noted, owing to the incomplete statement on the order of transitions into modulated phases, full marks were given for any satisfactorily justified answer for this part.) Part (c) was an application of the Gaussian approximation to computing a scaling exponent. Very few students gave a satisfactory answer here, since they failed to recognize that the scaling is modified at the Lifshitz point; extracting the exponent is a matter of dimensional analysis which is different at the Lifshitz point and away from it. Parts (d) and (e) focused on the case of a real scalar field. Unlike the complex case, the order parameter modulus fluctuates in space, necessitating an evaluation of an integral over each unit cell of the order, as suggested by a hint and the provision of the necessary integrals. (Failing to do the integral leaves an explicit spatial dependence that is tricky to manage) Students seemed to miss this interpretation, and so sensible answers to these two sections were rare. It was apparent that many students were struggling under time pressure owing to difficulties with part (b). This led to a low overall score for this question, around a third smaller than that expected, since it was anticipated that parts (a) and (b) would be high scoring for the majority of students.

**Note:** at least one student was confused by the statement that one should determine the free energies as functions of $\alpha_2$, $\rho_2$, $\rho_4 > 0$ and $\alpha_4 > 0$, taking the first inequality to apply to *all* the parameters rather than just $\rho_4$. This should have been clear from context of the problem (there would be only a single phase if all the parameters are bigger than zero) and the fact that inequalities were explicitly provided for $\alpha_4$ and $\rho_4$ separately. Students are reminded to carefully read the questions and to view the problems holistically, in which case such errors of interpretation would become evident.

**Collisionless Plasma Physics**

The standard of solutions returned by the candidates was generally good. All candidates attempted all questions. The difficulty of the questions appears to have been set at the correct level. A detailed report, question by question, is given below.

Question 1.

In part (a), few candidates could provide a complete ordering argument to obtain equations (3) and (4). Most candidates were able to follow the reasoning in part (b). In part (c), the candidates correctly used perpendicular force balance to obtain the result (6). In part (d), most candidates failed to notice that $R(r, \theta)$ should be expanded in the final magnetic field. Some candidates were unable to find $\psi_0$, after failing to notice that $dP/d\psi_0$, rather than $dP/dr$, was specified in the problem.

Question 2.

Parts (a) and (b) were answered well. Most candidates managed to sketch the form of the magnetic field in part (c), but a few candidates struggled with the qualitative and quantitative descriptions of the particle motion. Part (d) was answered well by candidates that understood the method of characteristics.

Question 3.

Part (a) was answered well by all candidates. Most candidates were able to find the correct basis vectors in part (b). In part (c), the candidates understood how to find the dispersion relation, valid for $x \neq 0$, although some made algebraic errors. Part (d) was answered well. In part (e), some candidates were unable to find the expression for $x_c/L$. All candidates managed to demonstrate that waves cannot reflect for $k_{y0} = 0$ and $\Omega_e \ll \omega$. Finally, in part (f), few candidates could correctly identify the range of $k_{y0}$ for which reflection occurs – the most common mistake was to neglect the fact that $k_{y0}$ is constrained by the X-mode dispersion relation at $x = 0$.

**Groups and Representations**

Overall, students performed very well on this paper and almost all of them have taken in the main messages of the course.

**Question 1:** The main difficulty was, in fact, part a), as students struggled to understand symmetric and anti-symmetric tensor products. Also part e) caused occasional conceptual difficulty. Part b), c) and d) were done confidently and correctly by almost all who attempted the question.

**Question 2:** While part a) and b) were, on the whole, done very well many students struggled with part c) which asked to find the two irreducible parts in a reducible representation explicitly. This often led to follow-on difficulties in the remainder of the question.

**Question 3:** The first four parts were completed very well and problems mostly arose in the last part e) when the result had to be applied to a proposed $G_2$ quark model.

**Question 4:** Part a) was done very well, but some students struggled to get to the correct branching for some cases in part b) which led to follow-on difficulties in part c). Few were able to apply to tensor transformation rules from part a) to complete part d) convincingly.

**Kinetic Theory**

Q1 Some candidates omitted the arguments of functions for brevity, notably in parts (b) and (e), and then confused themselves. Questions are written in a "show that" style to allow candidates to attempt later parts of a question from correct starting points. A full solution to a "show that" question should establish that the candidate could have derived the result without having been given it to aim for. A prose plausibility argument for the given result is not sufficient.

Part (a) was done well by all candidates.

Part (b) was found surprisingly difficult. $\phi(|\mathbf{x}_i - \mathbf{x}_j|)$ depends on both $\mathbf{x}_i$ and $\mathbf{x}_j$. Taking the time derivative gives two equal contributions to $\mathrm{d}E/\mathrm{d}t$

whose sum can be written as $-(1/N)\sum_{i=1}^{N}|\dot{\mathbf{x}}_i|^2$.

Parts (c) and (d) were mostly done well.

Part (e) was found most challenging, and not all candidates attempted it. The easiest approach begins with

$$\partial_t \rho_s^{(\infty)} = \frac{\partial}{\partial t}\left(\prod_{k=1}^{s}\rho_1^{(\infty)}(\mathbf{x}_k,t)\right) = \sum_{j=1}^{s}\left\{\frac{\partial}{\partial t}\rho_1^{(\infty)}(\mathbf{x}_j,t)\right\}\prod_{\substack{k=1\\k\neq j}}^{s}\rho_1^{(\infty)}(\mathbf{x}_k,t),$$

then uses the evolution equation for $\partial_t\rho_1^{(\infty)}(\mathbf{x}_j,t)$ with the second particle in the integral chosen to be particle $s+1$ using the particle exchange symmetry.

Q2 Performance on Question 2 was mediocre. Everyone did the standard bit — part (a) — more or less perfectly, competently reproducing the standard calculations for the hydro beam instabilities. In part (b), while most students figured out that the first and the third term in the dispersion relation would be the largest (and hence $ku_e \sim \omega_{pe}$), no one realised that it is from the balance of the *second* term and the (small) difference between the first and the third that the value of $p$ would actually be obtained

Part (c) should have been easy regardless of success in part (b), but mostly wasn't. In part (d), they all knew the instability was hydrodynamic but not all seemed to understand why or what that actually meant.

Q3

(a) There was some confusion which led some candidates to add a kinetic term in the discrete Hamiltonian. Here, as hinted in the definition of $E$, one simply has $H_\mathrm{d}(\mathbf{w})=U_\mathrm{ext}(\mathbf{w})+\int \mathrm{d}\mathbf{w}' U(\mathbf{w},\mathbf{w}')F(\mathbf{w}')$. Fortunately, this confusion did not prevent all the candidates from correctly deriving the Klimontovich equation.

(b) This question was only addressed by a small number of candidates. It did not present any particular difficulty, and the candidates correctly used the symmetry relation $U(\mathbf{w},\mathbf{w}')=U(\mathbf{w}',\mathbf{w})$.

(c) This question did not cause any difficulty.

(d) Many candidates forgot to mention that $\partial F_0/\partial t = -\left\langle\left[\delta F, \delta H\right]\right\rangle$, i.e. this term is second-order in the perturbations, so that it can be neglected in the system's first-order evolution equation.

(e) The manipulations of equations were done correctly, except, in some cases, for the explicit mention that having $\mathrm{Im}(\omega)>0$ large enough ensures the appropriate vanishing of $\mathrm{e}^{\mathrm{i}\omega t}\delta F(t)$ for $t\to+\infty$.

(f) This question has been adequately solved by all candidates.

(g) This question was only tackled by a small number of candidates.

**Quantum Field Theory**

Question 1: This question was mostly very well done. Candidates knew the Feynman rules and were able to apply them. The very last part on threshold behaviour produced quite a range of sensible comments but few candidates realised the implication that the width should vanish at $M = 3m$.

Question 2: This was the question candidates found hardest. Some could not use completeness of the spinor basis and failing to keep good account of the sign of three-momenta caused trouble in the last part. Few candidates recognised the canonical conjugate momentum to $\psi$.

Question 3: This question was well done on the whole. Some candidates did the question in four dimensions rather than three. Quite a few candidates got distracted with two vertex diagrams that only generate a mass correction (not asked for in the question).

**Quantum Matter**

This was a harder exam than average: mainly due to the last two parts of the first question, which involved some physical interpretation rather than calculations. Students found this difficult.

Q1.

(a) and (b) were done almost perfectly by almost everyone. (c) was supposed to be fairly easy, but in fact few people got it entirely correct. (d) and (e) which were relatively independendent of the earlier parts were meant to test physical understanding rather than calculation. These proved very challenging. Very few realized that the penetration length is crucial in (d). For part (e), I expected this part to be difficult, and it was. The point here was that Landau's argument is really asking about whether a moving system can give energy to the walls of the system or obstacles – whereas the galilean boost shown allows no such obstacles.

Q2.

This question was more straightforward, with one perfect score and several near perfect scores. The first two pieces were done almost perfectly by everyone. The third part was the main calculational part of the question.

While there were a wide range of minor errors and other confusions, a good fraction of the students basically went in the right direction.

### Radiative Processes and High Energy Astrophysics

#### Q1 (Radiative Processes)

**A:** Generally well answered. A few lost marks for not noting that the recombination lines result when the recombined electrons transition to lower bound states; instead saying that recombination lines result upon recombination (these are continuum photons).

**B:** Very well answered.

**C:** Generally well answered. The odd dropped mark for omitting the fator of 2.2 to convert from $\dot{N}_\gamma$ to $\dot{N}_{H_\alpha}$ and some dropped marks for not recognising that the Brackett series lines are less affected by dust than the $H_\alpha$ line.

**D:** The original question included the incorrect equation:

$$\Theta \equiv \frac{n_H \alpha_r (2\pi r_* D)^2 hc}{2.2 \langle\sigma\rangle F_{H_\alpha} \lambda_{H_\alpha}}.$$

The correct equation is

$$\Theta \equiv \frac{n_H \; \alpha_r \; r_*^2 \; hc}{8.8 \; \langle\sigma\rangle \; F_{H_\alpha} \; \lambda_{H_\alpha} \; D^2}.$$

All candidates attempted to derive the correct equation, and so saw the correction early enough. Most candidates recognised that the correct starting point is

$$\begin{aligned} \text{ionization rate per unit volume} &= \text{recombination rate per unit volume} \\ n_{HII} \int_0^\infty 4\pi\sigma_\nu \frac{I_\nu(r)}{h\nu} d\nu &= n_{HII}(r) n_e(r) \alpha_r. \end{aligned}$$

Candidates with lower marks from this part did not recognise that this was the starting point. The very best answers recalled that $I_\nu^* = F_\nu^*/\pi = L_\nu^*/(\pi \; 4\pi \; r_*^2)$. It is easy to introduce errors by losing a factor of 4 and/or using $r_{\rm in}$ instead of $r_*$.

**E:** The very best answers recognised that $I_\nu(r) = e^{-\tau_\nu} I_\nu^* \approx (1 - \tau_\nu) I_\nu^*$, where the second step is a Taylor expansion around $\tau_\nu = 0$. Some answers instead went straight to the second step. Candidates that scored poorly on this question did not recognise the starting point of part D.

**Q2 (High Energy Astrophysics)**

**A:** Generally well answered, especially the derivation.

**B-C:** Very well answered.

**D:** Very well answered, except some candidates did not remember the difference between the number density of *electrons*, $dN_e/dE \propto E^{-k}$, and the number density of *photons*, $dN_\gamma/d\nu \propto \nu^{-\alpha}$, where $\alpha = (k-1)/2$ for optically thin synchrotron radiation.

**E:** Generally well answered. Candidates that did not score as well did not realise that the maximum distance occurs when the jets start out at the speed of light and then decelerates.

### Supersymmetry and Supergravity

Question 1: Overall, the students performed well on part (a). The first bullet point was bookwork and all students gave good answers. The second bullet point presented some technical difficulties for some of the students, due to manipulations of spinor indices and related minus signs. Part (b) was bookwork and all students performed well. Part (c) was more challenging, as it was a variation on material discussed in the lectures. Roughly half of the students scored maximal or next-to-maximal points, with the other half attempting the problem and collecting a few points.

Question 2: Parts (a) and (b) were bookwork and everyone performed well. Most students got some points on part (c), even though only a few were able to get the full computation exactly right. Part (d) was a variation of examples discussed during class. Roughly half of the students got next-to-maximal points, while the others were able to collect a few marks.

### C3.3: Differentiable Manifolds

**Question 1.** Part (a) was mainly done well, though common errors were not mentioning that functions in the partition of unity subordinate to an atlas have support contained in charts in the atlas, and not commenting on the locally finite condition. Whilst many students obtained the right answers for (b), the required justifications were often lacking. Most students found part (c) challenging, only able to do the very first part, and otherwise unable to justify their answers fully for the other parts of the question. This was the most popular question and produced a wide range of marks from high to low.

**Question 2.** Part (a) was standard bookwork and typically answered correctly. Part (b)(i) was usually done well, though the most common difficulty was in showing that the map used to define $M$ as its zero set had 0 as a regular value. Students typically struggled with part (b)(ii), usually not attempting it and not finding the required map $g$ (though they realised it must be related to the determinant). Parts (c) and (d)(i) were again standard bookwork and usually answered correctly. Part (d)(ii) proved challenging for most students, with most getting stuck in length computations and not computing the preimage of the regular value correctly. There was a wide spread of marks for this question but there were no high marks (above 20).

**Question 3.** Part (a) was bookwork, but many students lost a mark by not mentioning the nondegeneracy of the canonical 2-form on the cotangent bundle in (a)(i). Part (b)(i) was usually done well, using a variety of methods. Part (b)(ii) proved to be challenging for the students, though most understood to use the known constants of the motion (such as the Hamiltonian), and several realised that it was useful to use Hamilton's equations. Most students had the right idea for both parts of (c), but quite often made computational errors. This was the least popular question and there was a wide range of marks from high to low.

### C3.4: Algebraic Geometry

All three questions elicited some very good answers, as well as a smaller number of weaker ones.

**Question 1**: This question was attempted by the overwhelming majority of candidates. Some students failed to see in (c) why the ideal is not prime, but most solutions cleared that hurdle. In (d), most students found the right form of the map, but several attempts at proving the surjectivity of this map were imprecise or incomplete, in particular including some handwaving around "choice of roots of unity". Some students proposed formulas for an inverse that included cube and other roots; such maps are clearly not regular in the sense the course defined regular maps. In (e), some students failed to address irreducibility; the easy argument that (d) implies that $C_2$ is a curve (so dim 1) was often missed. As for (f), most students failed to find the straightforward proof that the obvious 2-generator ideal is in fact prime.

**Question 2**: This question was attempted by about half the candidates. In (b), many students failed to notice that by differentiating $x^t Bx$, we get that the singular locus is simply the projectivisation of $\ker B$. Lots of essentially complete solutions were given to (c), either by direct calculation, or by recalling (in full) the Segre embedding. Part (d) was also done well by many students, either by explicitly factoring the equation or by writing down the equations of some line and solving equations for the coefficients. A comment that stated simply that ”this is a famous theorem about cubic surfaces” only received fractional credit, as this theorem was not part of the course.

**Question 3**: This question was also attempted by about half the candidates. Some students failed to note in (a) that essential aspects of a resolution of singularities are that it should be a surjective and a morphism. In (b), some students failed to address why the cover they propose is by affine varieties. (d) was generally done very well. In (e), some computational mistakes or failure to look at all affine charts suggested to some students that point blowup might actually lead to a resolution. (f) was generally done well unless time pressure prevented students from addressing this question.

### C5.6: Applied Complex Variables

Q1: This question was attempted by most candidates and was generally done well, apart from part (d) which required more independent thought. For Part (a), a few candidates were confused by interior and exterior angles, and in a number of cases the multiplicative constant $C$ in the mapping was left too general (or was assumed to be real). Parts (b) and (c) were managed fine by most candidates. For part (d), only a few gave a reasonably explanation of why the given quantity represents the film thickness at $C$. Most candidates realised they needed to do some sort of integral of the equations from (c), but many were confused by what limits to take.

Q2: This question was attempted by few candidates, perhaps reflecting the fact it looked most different from previous exam questions, although it followed a similar recipe. Parts (a) and (c) were both done well. The conversion between the limits as $Y \to \pm\infty$ and those as $z \to \infty$ and $z \to 0$ caused some difficulty in (b), and no-one really got very far with part (d). In particular, all but one attempt wrongly assumed that $H(z)$ needed to be zero.

Q3: This question was done very well on the whole, especially parts (a) and (b). For part (c) a common slip was to assume that $c > 0$ without comment, and in some cases the ordering of the logic to explain why the expression is constant was not quite right. For part (d), quite a number of candidates obtained the correct result, although the algebraic manipulations required a lot of reverse-engineering in some cases. The most common difficulties were errors in computing the residues, and having the wrong orientation of the inversion contour.

**C6.4: Finite Element Methods for Partial Differential Equations**

Q1: This question revealed a good spread of abilities across those who attempted it. Q1(a)(i) was answered correctly by every candidate. Some candidates claimed in Q1(a)(ii) that the property was immediate, neglecting to observe that $v \in \mathcal{P}_{k+1}(K)$. Q1(a)(iii) was answered correctly by every candidate. Q1(b)(i) was mostly answered well, with occasional slips in the signs of the integration by parts formula for curl. A few candidates erroneously applied the permutation rule for the scalar triple product directly to the volume integrand, rather than to the term arising via integration by parts. Q1(b)(ii) was mostly answered well. In Q1(b)(iii), a few candidates tried to modify the weak form they had already derived rather than starting from the strong form of the problem. While this approach can work, the handling of the boundary terms arising from integration by parts is delicate. A small number of candidates did not realise that $u \in H(\text{curl}, \Omega)$ does not imply that $\nabla \cdot u \in L^2(\Omega)$, and failed to integrate by parts appropriately to shift the differential operator onto the scalar-valued test function.

Q2: This question was very popular, with every candidate attempting it. Q2(a) was generally answered very well, as it was quite similar to problems seen on problem sheets. Q2(b) served useful in revealing a candidate's level of understanding. In Q2(c)(i), some candidates proposed a nonsymmetric but correct $A$, instead of the (obvious) symmetric $A$ that I had in mind. These candidates were awarded full marks, but generally ran into difficulties in handling the boundary conditions in Q2(c)(ii). Q2(c)(iii) was generally answered well, with most candidates giving sharp constants with the correct parametric dependence on $\beta$. Full marks were awarded for correct arguments leading to nonsharp bounds, so long as the dependence on $\beta$ (or not) was correct. In Q2(c)(iv), few candidates used the sharper $\sqrt{C/\alpha}$ bound available for symmetric problems, with several candidates erroneously claiming that the problem was not symmetric. The first part of Q2(c)(v) was generally answered well, but the latter part of characterising the kernel was only answered correctly by a handful of candidates.

Q3: (a)(i) was generally answered poorly, with few candidates getting the boundary integrals arising from integration by parts correct, and several including a dependence on the test function $v$ in the strong statement of the boundary condition $h(u, p, n) = 0$. Q3(a)(ii) was also answered poorly, with few candidates hitting the nail on the head: the boundary condition depends on the value of the pressure, so taking a solution $(u, p)$ and modifying it to $u, p + c)$ for $c \in \mathbb{R}$ no longer satisfies the boundary condition. Q3(a)(iii) was generally answered well, but several candidates ran out of steam in the calculations and claimed the result without showing it. Q3(a)(iv) was related to Q3(a)(i) and was marked in a manner so as not to penalise candidates for the same mistake twice. Q3(b)(i) was mostly answered well. Q3(b)(ii) was answered excellently, with most of the candidates who attempted it giving very clear arguments. Q3(b)(iii) appears to have been much more challenging, with only the best students answering it correctly. Several of these offered a beautiful argument based on applying Babuška's theorem.

### C3.2 Geometric Group Theory

Question 1 was attempted by all candidates, with good results on the whole. Most of the mistakes were in part c, where a number of attempts to change the presentations were made without any kind of method, and this turned out to be either unsuccessful or time consuming.

Question 2 was likewise popular. In the second part of (a) a number of candidates tried to use properties of universalities of amalgamated products, instead of an approach using actions on Bass-Serre trees. Surprisingly, the last question had the least number of successful attempts.

Question 3 was attempted by few candidates ,with only about a quarter of the questions receiving correct answers. This may be as usual related to the fact that this question covered the last part of the course.

### C7.5: General Relativity I

Question 1 was relatively popular, with a large majority of students attempting this question. The first part of the question and some portion of the middle of the question was successfully completed by many students, but there were a significant number of errors of understanding. It was fairly common, for example, to believe that a stationary observer follows a geodesic. Many students were able to obtain an expression for the time difference between the two observers, however in a large number of cases this expression included constants of motion which could (and should) have been eliminated. There were several ways to do this: the simplest was to use the stability of the orbit as in the lecture notes, but this was missed by a majority of students. Consequently they were unable to finish the question.

Question 2 was the most popular question, and part (a) was successfully completed by most students. Mistakes in part (b) were more common, with some students unable to correctly vary the action. Even fewer students succeeded in part (c): most students did not begin by finding the conserved quantities given by the Lagrangian, and many students mistakenly believed that the particle moves on a geodesic, despite deriving the force in part (b). Many students were more successful with the tensor algebra bit of part (d), although the majority of students did not realise that the energy-momentum tensor they had derived was anisotropic, and since the spacetime is isotropic this requires some extra matter.

Question 3 was the least popular question. Part (a) was done successfully by the majority of students, and a good number of students did well in part (b) too, although some struggled with the tensor algebra. Many also struggled with the tensor algebra in part (c). In part (d), no student gave a satisfactory argument for $\sigma^2 \geq 0$, although some did notice that such an argument was needed. The interpretation of the result in part (e) was also mistaken in a large number of solutions.

### C7.6: General Relativity II

**Question 1**: The first question was attempted by nearly all students. The tensor manipulations in part a) did not prove difficult and also part b) was executed nearly flawlessly. However, part c) was already more difficult and although most students scored a good number of points, only a few carried it out more or less correctly. Question d) i) was the most difficult part and was not answered correctly by any student. Part ii), however, was again easier with a few students scoring full marks and most students at least some.

**Question 2**: This question was the least popular with only a few students attempting it. Part a) was carried out well by everyone and in part b) everyone showed the equivalence of the gravitational perturbations under the gauge transformation, however, only half the students showed that in general the new perturbation is not in wave gauge. Part c) and d) proved difficult and no student produced a complete answer here.

**Question 3**: This question was again attempted by nearly all students. Part a) was executed flawlessly by nearly everyone. Part b) was solved very well by most students and various computational routes to the correct solution were presented. The last part was the most difficult, and while most students gained a few points here, no one delivered a complete solution. In particular students struggled with keeping track of the coordinate ranges and with constructing the maximal analytic extension.

**C7.4: Introduction to Quantum Information**

Question 1. It was the least popular question. Some students struggled with visualising mixed states in terms of Bloch vectors and failed to draw the required mixture of $Z$ eigenstates in (a) and the mixture of Pauli states in (b). Those who failed to sketch the octahedron in part (b) had difficulties with completing part (e), which is related to (b). Only few students attempted parts (e) and (f). Those who attempted part (f) showed no difficulty in stepping through the quantum circuit. However, many students simply stated that the $T$ gate is needed for universality without backing it up with any arguments.

Question 2. It was by far the most popular, with nearly all students having it used as a successfully attempted question. In general, it was very well answered, and students scored well. Part (a) was bookwork; part (b) was done in a few different ways, but almost always successfully; part (c) was where most students dropped a few marks, having calculated the required identity successfully, but then incorrectly assuming that they could simply square this to obtain the probability; part (d) was well answered, but many students simply listed four probabilities without even mentioning how they related to the actual question asked; part (e) was usually answered correctly, and many even mentioned how the maximally mixed state attains the maximal probability for this test; part (f) was usually either answered entirely correctly, or entirely incorrectly, with some students simply stating that the probability obtained in part (d) was independent of the random number generator.

Question 3. The first three parts of this question (a, b and c) were, in general, well answered. Students knew how to take partial traces, construct the Choi matrix, and how to check positivity and complete positivity of linear maps. Some students did not realise that in part (b) the reduced density operators must have the same spectrum. The last two parts (d and e) — which required visualising the action of the depolarising map on the Bloch vector — turned out to be the most challenging; students tried different approaches but only two of them (out of 27) got it right.

**C3.5: Lie Groups**

Candidates found this a hard paper, though there were some good answers, especially to Question 1.

All candidates attempted Question 1. The part which caused most difficulty was 1(a)(iii): very few candidates were able to describe the irreducible representations of $O(2)$ correctly. Many forgot to include the hypothesis that $G$ should be connected in the statement of the Maximal Torus Theorem required in 1(b), though it appeared that some realised the relevance of connectedness when answering 1(d) and then corrected their answers to 1(b).

The amount of unseen material in Question 2 was perhaps off-putting to candidates. There were some nice answers to 2(a) and 2(b) but almost no attempts at (c), possibly from lack of time.

More candidates attempted Question 3. On the whole 3(a) and 3(c) were well done, although there were not many completely satisfactory descriptions of the integrand in the Weyl integration formula. 3(b) was found much harder, especially 3(b)(iii), even though there was a similar calculation in the lecture notes.

**C5.5 Perturbation Methods**

*Q1*

Overall the question was answered well. The first part of the question presented little difficulty in general though a small number of candidates stopped at the first iteration, thus failing to confirm that the first iterate was indeed the first term in an asymptotic expansion. In the second part there were numerous good solutions though justifying that the infinite number of corrections, even when summed, were still $o(1/x^m)$ distinguished the best solutions.

*Q2*

This was the least popular question, though it was a generalisation of a very similar problem in the lecture notes. The latter entails that students trying this question as part of revision in future years will find it more difficult than intended if the lectures no longer consider this example. The fundamental difference with the lectured example was the $\exp(xu)$ term, which could be expanded via

$$\exp(xu) = \exp(xu_0 + \epsilon xu_1 + \ldots)$$

after which the structure of the question was similar, though the complexities of dealing with the $f(x, x/\epsilon)$ term, which had no analogue in lectures, differentiated the best candidates.

*Q3*

This was the most popular question and it was answered well in general. Most solutions picking up all or most of marks controlled the complexity by noting that using the expression in part (a), together with the equations governing $\varphi$ and $A_0$ in part (b), gave an extensive simplification for

$$\frac{\mathrm{d}^3 y_W}{\mathrm{d}x^3} + x^2 y_W.$$

## E. Comments on performance of identifiable individuals

Removed from public version.

## F. Names of members of the Board of Examiners

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Prof Artur Ekert (Mathematical Institute, University Of Oxford)
Prof Lionel Mason (Mathematical Institute, University Of Oxford)
Dr Dmitri Pushkin (Department of Mathematics, University of York)
Prof Alex Schekochihin (Department of Physics, University of Oxford; Chair)
Dr David Skinner (Department of Applied Mathematics and Theoretical Physics, University of Cambridge)
Prof Subir Sarkar (Department of Physics, University of Oxford)

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Prof Michael Barnes
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Dr Cyril Closset
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