Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

KINETIC THEORY

Hilary Term 2021

THURSDAY, 14TH JANUARY 2021, Opening Time: 09:30 a.m GMT

You should submit answers to all three questions.

You have **3 hours** *writing time to complete the paper and up to* **1 hour** *technical time for uploading your file. The allotted technical time must not be used to finish writing the paper.*
You are permitted to use the following material(s):
Calculator
The use of computer algebra packages is **not** *allowed.*

1. This question concerns a population of indistinguishable particles interacting through a pairwise potential, while also losing momentum through collisions with an otherwise inert background medium. The pairwise interactions are weak and long-range with lengthscale L. The particles have positions $\mathbf{X}_i$ and velocities $\mathbf{V}_i$. They all have mass m. Their equations of motion are

$$\dot{\mathbf{X}}_i = \mathbf{V}_i, \quad m\dot{\mathbf{V}}_i = -\gamma \mathbf{V}_i + \frac{F_0}{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \mathbf{F}\left(\frac{\mathbf{X}_i - \mathbf{X}_j}{L}\right),$$

where $\mathbf{F}$ is a dimensionless function of a dimensionless argument. A dot denotes a derivative with respect to dimensional time. The sum is taken over $j = 1, 2, \ldots, N$ but omits the self-interaction term with $j = i$. The constant F_0 sets the strength of the inter-particle forces, and the constant γ sets the rate of momentum loss through collisions with the background.

(a) [4 marks] By writing $\mathbf{X}_i = L\mathbf{x}_i$ and introducing a suitable dimensionless time variable t, derive the dimensionless equation

$$\epsilon \frac{\mathrm{d}^2 \mathbf{x}_i}{\mathrm{d}t^2} + \frac{\mathrm{d}\mathbf{x}_i}{\mathrm{d}t} = \frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \mathbf{F}(\mathbf{x}_i - \mathbf{x}_j),$$

and give an expression for the dimensionless parameter ϵ.

For the rest of this question we will set $\epsilon = 0$, and assume that the force is derived from a potential:

$$\mathbf{F}(\mathbf{x}_i - \mathbf{x}_j) = -\frac{\partial \phi(|\mathbf{x}_i - \mathbf{x}_j|)}{\partial \mathbf{x}_i}.$$

(b) [4 marks] Show that the energy of the system,

$$E(t) = \frac{1}{2N^2} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \phi(|\mathbf{x}_i - \mathbf{x}_j|),$$

cannot increase over time.

(c) [5 marks] Now consider an ensemble of such systems described by a probability density function $\rho(\mathbf{x}_1, \ldots, \mathbf{x}_N, t)$ on a $3N$-dimensional phase space. What symmetry property should ρ satisfy?

By considering the expected number of particles in some fixed volume Ω of phase space, derive the Liouville equation

$$\frac{\partial \rho}{\partial t} + \frac{1}{N} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{\partial}{\partial \mathbf{x}_i} \cdot (\mathbf{F}(\mathbf{x}_i - \mathbf{x}_j)\rho) = 0.$$

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(d) [5 marks] Show that the reduced s-particle probability density functions $\rho_s^{(N)}(\mathbf{x}_1, \ldots, \mathbf{x}_s, t)$ for an ensemble of systems of N particles evolve according to the BBGKY hierarchy

$$\frac{\partial}{\partial t}\rho_s^{(N)} + \frac{1}{N}\sum_{i=1}^{s}\sum_{\substack{j=1\\ j\neq i}}^{s}\frac{\partial}{\partial \mathbf{x}_i}\cdot\left(\mathbf{F}(\mathbf{x}_i - \mathbf{x}_j)\rho_s^{(N)}\right)$$
$$+\frac{N-s}{N}\sum_{i=1}^{s}\int \mathrm{d}\mathbf{x}_{s+1}\,\frac{\partial}{\partial \mathbf{x}_i}\cdot\left(\mathbf{F}(\mathbf{x}_i - \mathbf{x}_{s+1})\rho_{s+1}^{(N)}(\mathbf{x}_1, \ldots, \mathbf{x}_{s+1}, t)\right) = 0.$$

(e) [7 marks] Now consider the limit in which the number of particles N goes to infinity. Show that the corresponding limit of the BBGKY hierarchy allows factorised solutions of the form

$$\rho_s^{(\infty)}(\mathbf{x}_1, \ldots, \mathbf{x}_s, t) = \prod_{i=1}^{s}\rho_1^{(\infty)}(\mathbf{x}_i, t).$$

What property of the system does this represent?

[Hint: you may wish to try the case with $s = 2$ first before attempting the general case.]

2. Consider a plasma consisting of Maxwellian ions (mass m_i, charge $q_i = Ze$, density n_i, temperature T_i, thermal speed $v_{\text{th}i} = \sqrt{2T_i/m_i}$) and a cold electron beam (mass m_e, charge $q_e = -e$, density n_e, velocity $u_e \gg v_{\text{the}}$—much greater than the width of the distribution).

   (a) [7 marks] Starting from the standard expression for the plasma dielectric function describing infinitesimal electrostatic perturbations with wave vector k in the direction of the beam and $\propto e^{pt}$ (where t is time and p is, in general, complex), assume $|p/k| \gg v_{\text{th}i}$ and show that the dispersion relation is

$$1 + \frac{\omega_{\text{p}i}^2}{p^2} - \frac{\omega_{\text{pe}}^2}{(ku_e - ip)^2} = 0, \tag{1}$$

   where $\omega_{\text{p}i}$ and ω_{pe} are the ion and electron plasma frequencies, respectively.

   (b) [7 marks] Looking for solutions with $\omega_{\text{pe}} \gg |p| \gg \omega_{\text{p}i}$, show that there is an instability that attains its maximum growth rate at $k = \omega_{\text{pe}}/u_e$.

   Hint. You can do this either by identifying the dominant balance in the dispersion relation or by looking for a solution in the form $p = |p|e^{i\theta}$ and maximising the growth rate with respect to θ.

   (c) [7 marks] Show that the maximum growth rate is

$$\gamma = \frac{\sqrt{3}}{2^{4/3}}\,\omega_{\text{pe}}^{1/3}\omega_{\text{p}i}^{2/3}. \tag{2}$$

   (d) [4 marks] Is this instability kinetic or hydrodynamic? Do Landau resonances with either ions or electrons play a role? Explain your reasoning.

3. We consider a Hamiltonian system in a phase space of dimension $2d$ with the canonical coordinates $\mathbf{w} = (\mathbf{q}, \mathbf{p})$. The system is composed of N particles each of mass m, embedded within an external potential $U_{\mathrm{ext}}(\mathbf{w})$ and coupled to one another via the long-range pairwise symmetric interaction potential $U(\mathbf{w}, \mathbf{w}')$.

   (a) [4 marks] The system's instantaneous state is described by the distribution function (DF)

$$F_{\mathrm{d}}(\mathbf{x}, \mathbf{v}, t) = \sum_{i=1}^{N} m\, \delta_{\mathrm{D}}(\mathbf{w} - \mathbf{w}_i(t)). \tag{3}$$

   Show that F_{d} evolves according to

$$\frac{\partial F_{\mathrm{d}}}{\partial t} + \big[F_{\mathrm{d}}, H_{\mathrm{d}}\big] = 0, \tag{4}$$

   and give expressions for the Hamiltonian H_{d} and the operator $[\,\cdot\,,\,\cdot\,]$.

   (b) [4 marks] Show that Eq. (4) exactly conserves the total energy

$$E(t) = \int \mathrm{d}\mathbf{w}\, U_{\mathrm{ext}}(\mathbf{w})\, F_{\mathrm{d}}(\mathbf{w}, t) + \frac{1}{2}\int \mathrm{d}\mathbf{w}\mathrm{d}\mathbf{w}'\, U(\mathbf{w}, \mathbf{w}')\, F_{\mathrm{d}}(\mathbf{w}, t)\, F_{\mathrm{d}}(\mathbf{w}', t). \tag{5}$$

   (c) [4 marks] The system's mean-field state is described by $F_0 = \langle F_{\mathrm{d}} \rangle$ and $H_0 = \langle H_{\mathrm{d}} \rangle$. State the meaning of the symbol $\langle\,\cdot\,\rangle$. Assume that the system is in an integrable equilibrium associated with some angle-action coordinates $(\boldsymbol{\theta}, \mathbf{J})$. Give two properties of angle-action coordinates. What can you say about F_0 and H_0 in angle-action coordinates?

   (d) [2 marks] Instantaneous perturbations in the system's DF and Hamiltonian are denoted as $\delta F(\mathbf{w}, t)$ and $\delta H(\mathbf{w}, t)$. Write down the evolution equation of δF at first order in the perturbations in angle-action coordinates.

   (e) [3 marks] The Laplace-Fourier transform of any $F(\boldsymbol{\theta}, \mathbf{J}, t)$ is defined to be

$$\widehat{F}_{\mathbf{k}}(\mathbf{J}, \omega) \equiv \int_0^{+\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} \int \frac{\mathrm{d}\boldsymbol{\theta}}{(2\pi)^d}\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\boldsymbol{\theta}}\, F(\boldsymbol{\theta}, \mathbf{J}, t), \tag{6}$$

   with $\mathbf{k} \in \mathbb{Z}^d$. Show that the evolution equation for δF can be recast as

$$\delta\widehat{F}_{\mathbf{k}}(\mathbf{J}, \omega) = -\frac{\mathbf{k}\cdot\partial F_0/\partial \mathbf{J}}{\omega - \mathbf{k}\cdot\boldsymbol{\Omega}(\mathbf{J})}\, \delta\widehat{H}_{\mathbf{k}}(\mathbf{J}, \omega) - \frac{\delta F_{\mathbf{k}}(\mathbf{J}, 0)}{\mathrm{i}(\omega - \mathbf{k}\cdot\boldsymbol{\Omega}(\mathbf{J}))}, \tag{7}$$

   and write down an expression for $\boldsymbol{\Omega}(\mathbf{J})$ that appears in this equation.

   (f) [4 marks] In the long-term, the perturbations present in the system drive the relaxation of the system's overall orbital structure. Show that it is governed by

$$\frac{\partial F_0(\mathbf{J}, t)}{\partial t} = -\frac{\partial}{\partial \mathbf{J}} \cdot \left[\int \frac{\mathrm{d}\boldsymbol{\theta}}{(2\pi)^d} \left\langle \frac{\partial \delta F}{\partial \boldsymbol{\theta}}\, \delta H \right\rangle\right]. \tag{8}$$

   (g) [4 marks] Assume that a second population of particles of individual mass m' described by the DF $G_{\mathrm{d}}(\boldsymbol{\theta}, \mathbf{J}, t)$ also orbits within the system. How does this change H_{d}? Stating clearly all your assumptions, show that both populations undergo an orbital relaxation described by

$$\begin{aligned} \frac{\partial F_0(\mathbf{J}, t)}{\partial t} &= -\frac{\partial}{\partial \mathbf{J}} \cdot \left[m\, \mathbf{D}_1(\mathbf{J})\, F_0(\mathbf{J}, t) - \mathbf{D}_2(\mathbf{J})\cdot \frac{\partial F_0}{\partial \mathbf{J}}\right], \\ \frac{\partial G_0(\mathbf{J}, t)}{\partial t} &= -\frac{\partial}{\partial \mathbf{J}} \cdot \left[m'\, \mathbf{D}_1(\mathbf{J})\, G_0(\mathbf{J}, t) - \mathbf{D}_2(\mathbf{J})\cdot \frac{\partial G_0}{\partial \mathbf{J}}\right]. \end{aligned} \tag{9}$$

   [Explicit expressions for $\mathbf{D}_1(\mathbf{J})$ and $\mathbf{D}_2(\mathbf{J})$ are **not** required.] Give one example of an astrophysical system against which these equations might usefully be tested.