Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

KINETIC THEORY

Hilary Term 2018

THURSDAY, 11TH JANUARY 2018, 9:30am to 12:30pm

You should submit answers to all three questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. Consider a Hamiltonian system of N particles of unit mass interacting through a pairwise potential ϕ. Any function F of the particle positions $\mathbf{x}_i$, velocities $\mathbf{v}_i$, and time evolves according to

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F, H\},$$

where

$$H = \frac{1}{2}\sum_{i=1}^{N} |\mathbf{v}_i|^2 + \sum_{1\leqslant i<j\leqslant N} \phi(|\mathbf{x}_i - \mathbf{x}_j|),$$

and

$$\{A, B\} = \sum_{i=1}^{N} \left(\frac{\partial A}{\partial \mathbf{x}_i} \cdot \frac{\partial B}{\partial \mathbf{v}_i} - \frac{\partial B}{\partial \mathbf{x}_i} \cdot \frac{\partial A}{\partial \mathbf{v}_i} \right).$$

(a) [3 marks] Write down the evolution equation for the N-particle density ρ. Define the 1-particle and 2-particle densities ρ_1 and ρ_2, and the distribution functions f_1 and f_2. Give a brief justification for the definitions of the latter.

(b) [10 marks] By rewriting the Hamiltonian as a sum of three terms, or otherwise, show that $f_1(\mathbf{x}_1, \mathbf{v}_1, t)$ evolves according to

$$\frac{\partial f_1}{\partial t} + \{f_1, H_1\} = \int dV_2 \, \frac{\partial f_2}{\partial \mathbf{v}_1} \cdot \frac{\partial \phi(|\mathbf{x}_1 - \mathbf{x}_2|)}{\partial \mathbf{x}_1},$$

where H_1 is the single-particle Hamiltonian, and $dV_i = d\mathbf{v}_i d\mathbf{x}_i$.

(c) [6 marks] Now suppose that the two-particle distribution factorises as

$$f_2(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, t) = f_1(\mathbf{x}_1, \mathbf{v}_1, t) f_1(\mathbf{x}_2, \mathbf{v}_2, t).$$

Give an interpretation of this condition.

Show that $f(\mathbf{x}, \mathbf{v}, t) = f_1(\mathbf{x}_1, \mathbf{v}_1, t)$, with $_1$ suffices omitted for brevity, obeys the evolution equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = \nabla_{\mathbf{v}} \Phi \cdot \nabla_{\mathbf{v}} f, \qquad (\dagger)$$

and find an expression for Φ.

Show that ($\dagger$) has steady solutions of the form

$$f(\mathbf{x}, \mathbf{v}, t) = F(\tfrac{1}{2}|\mathbf{v}|^2 + G(\mathbf{x})),$$

and find an expression for G. Give a brief interpretation of this result.

(d) [6 marks] Derive an evolution equation for

$$H_B(\mathbf{x}, t) = \int d\mathbf{v} \, f \log f,$$

and comment on your result.

Derive evolution equations for the fluid mass and momentum densities associated with f.

2. Consider a population of particles of charge q and mass m. Assume that collisions are entirely negligible. Assume further that an electrostatic fluctuation field $\mathbf{E} = -\nabla\varphi$ (with zero spatial mean) is present and that this field is given and externally determined, i.e., it is unaffected by the particles that are under consideration. This might happen physically if, for example, the particles are a low-density admixture in a plasma consisting of some more numerous species of ions and electrons, which dominate the plasma's dielectric response.

   (a) [2 marks] Write down without derivation the evolution equation for the particles' distribution function $f(t, \mathbf{r}, \mathbf{v})$. Explain very briefly what physics the different terms represent.

   (b) [3 marks] Assume that the distribution function can be represented as $f = f_0(t, \mathbf{v}) + \delta f(t, \mathbf{r}, \mathbf{v})$, where the equilibrium distribution f_0 is spatially homogeneous and changes slowly in time compared to the perturbed distribution $\delta f \ll f_0$. Show that the evolution of the equilibrium distribution is described by

$$\frac{\partial f_0}{\partial t} = -i\frac{q}{m}\frac{\partial}{\partial \mathbf{v}} \cdot \sum_{\mathbf{k}} \mathbf{k}\, \langle \varphi_{\mathbf{k}}^* \delta f_{\mathbf{k}} \rangle\,, \tag{1}$$

   where angle brackets denote time average over the fast variation of the fluctuation field and $\varphi_{\mathbf{k}}$ and $\delta f_{\mathbf{k}}$ are the Fourier-transformed fields defined via

$$\varphi(t, \mathbf{r}) = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \varphi_{\mathbf{k}}(t), \quad \delta f(t, \mathbf{r}, \mathbf{v}) = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \delta f_{\mathbf{k}}(t, \mathbf{v}). \tag{2}$$

   (c) [7 marks] Assume that φ is sufficiently small for it to be possible to determine δf from the linearised kinetic equation. Let $\delta f = 0$ at $t = 0$. Show that f_0 satisfies a quasilinear diffusion equation

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v_i} D_{ij} \frac{\partial f_0}{\partial v_j}, \tag{3}$$

   where the diffusion matrix is

$$D_{ij} = \frac{q^2}{m^2} \sum_{\mathbf{k}} k_i k_j \frac{1}{2\pi i} \int \mathrm{d}p \, \frac{1}{p + i\mathbf{k}\cdot\mathbf{v}} \int_{-\infty}^{t} \mathrm{d}\tau \, e^{p\tau} C_{\mathbf{k}}(\tau), \tag{4}$$

   where the p integration is along a contour appropriate for an inverse Laplace transform and $C_{\mathbf{k}}(t - t') = \langle \varphi_{\mathbf{k}}^*(t) \varphi_{\mathbf{k}}(t') \rangle$ is the correlation function of the fluctuation field (which is taken to be statistically stationary, so $C_{\mathbf{k}}$ depends only on the time difference $t - t'$).

   (d) [7 marks] Let $C_{\mathbf{k}}(\tau) = A_{\mathbf{k}} e^{-\gamma_{\mathbf{k}}|\tau|}$ (i.e., $\gamma_{\mathbf{k}}^{-1}$ is the correlation time of the fluctuation field and $A_{\mathbf{k}}$ its spectrum; assume $\gamma_{-\mathbf{k}} = \gamma_{\mathbf{k}}$). Do the integrals in (4) and show that, at $t \gg \gamma_{\mathbf{k}}^{-1}$,

$$D_{ij} = \frac{q^2}{m^2} \sum_{\mathbf{k}} k_i k_j \frac{\gamma_{\mathbf{k}} A_{\mathbf{k}}}{\gamma_{\mathbf{k}}^2 + (\mathbf{k}\cdot\mathbf{v})^2}. \tag{5}$$

   (e) [6 marks] Restrict consideration to one spatial dimension and to the limit in which $\gamma_k \gg kv$ for typical wave numbers of the fluctuations and typical particle velocities (i.e., the fluctuation field is short-time correlated). Assuming that f_0 at $t = 0$ is a Maxwellian with temperature T_0, predict the evolution of f_0 with time. Discuss what physically is happening to the particles. Discuss the validity of the short-correlation-time approximation and of the assumption of slow evolution of f_0. What is, roughly, the condition on the amplitude and the correlation time of the fluctuation field that makes these assumptions compatible?

3. (a) [7 marks] N point particles of mass m move in their collectively generated gravitational field. Given that the system's properties change negligibly on a dynamical timescale, show that $2T + W = 0$, where

$$T = \frac{1}{2}m\sum_{\alpha=1}^{N}|\dot{\mathbf{x}}_\alpha|^2; \quad W = -\frac{1}{2}Gm^2\sum_{\alpha\neq\beta=1}^{N}\frac{1}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|}.$$

(b) [3 marks] Use this relation to estimate the typical random velocity of stars in a cluster given that the cluster has a characteristic size $R = 5\,\mathrm{pc} \simeq 1.5 \times 10^{17}\,\mathrm{m}$ and assuming that all the cluster's 10^5 stars have mass $m = 0.8\,M_\odot \simeq 1.6 \times 10^{30}\,\mathrm{kg}$. Hence estimate the cluster's dynamical timescale commenting on the value you obtain.

(c) [5 marks] Why must the velocity distribution at any point in the cluster differ significantly from a Maxwellian? Why does this difference limit the possible lifetime of the cluster?

(d) [10 marks] Explain the concept of the *mean-field model* of a stellar system and state Jeans' theorem. Write down an equation satisfied by the distribution function (DF) f of the full-field model. By considering the fluctuations $f_1(\mathbf{x}, \mathbf{v}, t)$ in the actual DF that occur on a dynamical time around the mean-field DF f_0 and the associated fluctuation in the gravitational potential

$$\Phi_1(\mathbf{x}, t) = -G\int \mathrm{d}^3\mathbf{x}'\mathrm{d}^3\mathbf{v}\,\frac{f_1(\mathbf{x}, \mathbf{v}, t)}{|\mathbf{x}' - \mathbf{x}|}$$

show that f_0 evolves slowly according to the equation

$$\frac{\mathrm{d}f_0}{\mathrm{d}t} = -\langle[f_1, \Phi_1]\rangle,$$

explaining fully the meaning of the symbols on the right.