A15089W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

KINETIC THEORY

Hilary Term 2017

THURSDAY, 12th JANUARY 2017, 09:30am to 12:30pm

You should submit answers to all three questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. The Boltzmann equation for a function $f(\mathbf{x}, \mathbf{v}, t)$ with the BGK model collision operator is

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f = -\frac{1}{\tau}\left(f - f^{(0)}\right), \text{ where } f^{(0)} = \frac{\rho}{(2\pi\theta)^{3/2}}\exp\left(-\frac{|\mathbf{v}-\mathbf{u}|^2}{2\theta}\right),$$

and τ is a positive constant. The particles may be assumed to have unit mass.

(a) [6 marks] Give physical interpretations of the quantities ρ, $\mathbf{u}$, θ. Explain how they are calculated in the BGK collision operator, and derive macroscopic conservation laws for the mass, momentum, and energy densities. How is the energy density related to θ?

(b) [7 marks] The peculiar velocity is $\mathbf{w} = \mathbf{v} - \mathbf{u}(\mathbf{x}, t)$. What is its physical interpretation? By considering a solution of the Boltzmann–BGK equation in the form

$$f(\mathbf{x}, \mathbf{v}, t) = F(\mathbf{x}, \mathbf{v} - \mathbf{u}(\mathbf{x}, t), t),$$

or otherwise, show that the Boltzmann–BGK equation becomes (in Einstein summation convention)

$$\frac{\partial F}{\partial t} + (u_i + w_i)\frac{\partial F}{\partial x_i} - \left(\frac{\partial u_j}{\partial t} + (u_i + w_i)\frac{\partial u_j}{\partial x_i}\right)\frac{\partial F}{\partial w_j} = -\frac{1}{\tau}\left(F - F^{(0)}\right), \tag{1}$$

when $\mathbf{x}$, $\mathbf{w}$, t are now treated as independent variables, and give an expression for $F^{(0)}$.

(c) [12 marks] Suppose that $\mathbf{u}(\mathbf{x}, t) = \mathbf{x}\cdot\mathsf{A}$, with A a constant matrix, describes a steady, incompressible shear flow.

Show that spatially homogeneous solutions of (1) satisfy

$$\frac{\partial F}{\partial t} - w_i A_{ij}\frac{\partial F}{\partial w_j} = -\frac{1}{\tau}\left(F - F^{(0)}\right).$$

Verify that these solutions are compatible with mass and momentum conservation. Show further that the internal energy density $\epsilon = (1/2)\int \mathrm{d}\mathbf{w}\,|\mathbf{w}|^2 F$ satisfies an equation of the form

$$\frac{\partial \epsilon}{\partial t} = -A_{ij}P_{ij},$$

and define P_{ij}.

Give a physical interpretation of this equation. Show that the right-hand side is non-negative when P_{ij} takes the form that leads to the Navier–Stokes equations.

2. Consider a plasma consisting of electrons and ions in a one-dimensional, spatially homogeneous, constant in time, static (no mean flows) Maxwellian equilibrium. Allow infinitesimal perturbations of the electron distribution function, $f = f_0 + \delta f$, while assuming that the ion equilibrium is unperturbed. Assume also small electric perturbations $E = -\partial\varphi/\partial x$ and no magnetic fields, either equilibrium or perturbed.

   (a) [4 marks] Starting from the linearised, collisionless kinetic equation for the perturbed electron distribution function, show that the perturbed electron density $\delta n = \int \mathrm{d}v\, \delta f$ and perturbed electron flow velocity $u = (1/n_0)\int \mathrm{d}v\, v\delta f$ (where n_0 is the equilibrium electron density) satisfy the following "hydrodynamic" equations

$$\frac{\partial \delta n}{\partial t} + \frac{\partial}{\partial x} n_0 u = 0, \tag{2}$$

$$m n_0 \frac{\partial u}{\partial t} = -\frac{\partial \delta p}{\partial x} - e n_0 E, \tag{3}$$

   where m is the electron mass, e the elementary charge and δp is the perturbed electron pressure (provide its definition).

   (b) [5 marks] Assume that the electrons have the "adiabatic" equation of state

$$p n^{-\Gamma} = \text{const}, \tag{4}$$

   where $p = p_0 + \delta p$ and $n = n_0 + \delta n$ are their pressure and density, respectively, and Γ is some exponent. Hence derive from equations (2–3) the dispersion relation for the waves that they support. Give a physical interpretation of these waves. At this stage, what do you expect should be the value of Γ in order for your result to agree with kinetic theory? [Some useful definitions: electron thermal speed $v_{\text{th}} = \sqrt{2T_0/m}$, Debye length $\lambda_{\text{D}} = v_{\text{th}}/\sqrt{2}\omega_{\text{p}}$, where ω_{p} is the electron plasma frequency, T_0 equilibrium electron temperature.]

   (c) [9 marks] Going back to the linearised 1D kinetic equation and assuming an initial perturbation with wave number k, show that the perturbed distribution function is

$$\delta f_k(t) = -\frac{e}{m}\varphi_k(t)\frac{1 - e^{-i(kv-\omega_k)t-\gamma_k t}}{kv - \omega_k - i\gamma_k} k \frac{\partial f_0}{\partial v} + g_k e^{-ikvt}, \tag{5}$$

   where g_k is the initial perturbation and $\omega_k + i\gamma_k$ is the complex frequency of the oscillating mode, i.e., $\varphi_k(t) \propto e^{-i\omega_k t + \gamma_k t}$. Comment on what various terms in (5) represent and how they vary with time. Without derivation, explain physically why the waves are damped ($\gamma_k < 0$), and why $\gamma_k \ll \omega_k$ if $\omega_k \gg k v_{\text{th}}$.

   (d) [7 marks] Assuming $\omega_k \gg k v_{\text{th}}$ (equivalently, $k\lambda_{\text{D}} \ll 1$), and using (5), calculate δn and δp due to nonresonant (thermal-bulk) particles. Hence show that $\Gamma = 3$, i.e., that electrons in a Langmuir wave behave like a 1D adiabatic fluid.

   [Before doing the calculation, argue why the e^{-ikvt} terms can be dropped in the calculation of velocity integrals, at large t. Could they have been dropped had we wanted to calculate density and pressure perturbations due to the resonant particles with velocities near $v = \omega_k/k$?]

   Why was the damping of Langmuir waves not captured by equations (2–4)?

3. (a) [10 marks] An equilibrium stellar system is described by the distribution function (DF) $f_0(\mathbf{x}, \mathbf{v})$ and the mean-field potential $\Phi_0(\mathbf{x})$. Write down two equations that must be satisfied by f_0 and Φ_0.

   Let $f_1(\mathbf{x}, \mathbf{v}, t)$ be the small change in the system's DF when it is out of equilibrium. Obtain the equation that governs the evolution of f_1 to first order in small quantities.

   State three properties of angle-action coordinates $(\boldsymbol{\theta}, \mathbf{J})$. Use these coordinates to simplify your equation for f_1.

   (b) [11 marks] Fluctuations in the DF cause the equilibrium state f_0 to evolve slowly. Show that this evolution is governed by the equation

$$\frac{\partial f_0}{\partial t} = -\mathrm{i}\frac{\partial}{\partial \mathbf{J}} \cdot \left\langle \sum_{\mathbf{n}} \mathbf{n} \hat{f}_1(\mathbf{n}, \mathbf{J}, t) \hat{\Phi}_1(-\mathbf{n}, \mathbf{J}, t) \right\rangle,$$

   where the hat operator is such that

$$\hat{g}(\mathbf{n}) \equiv \int \frac{\mathrm{d}^3\boldsymbol{\theta}}{(2\pi)^3}\, g(\boldsymbol{\theta})\, \mathrm{e}^{-\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}}.$$

   Show that the right side of the equation for $\partial f_0/\partial t$ is real and explain the significance of its taking the form of a divergence.

   (c) [4 marks] The evolution equation can be brought to the form

$$\frac{\partial f_0}{\partial t} = -\frac{\partial}{\partial \mathbf{J}} \cdot \left(\mathbf{D}_1(\mathbf{J}) f_0 + \mathbf{D}_2(\mathbf{J}) \cdot \frac{\partial f_0}{\partial \mathbf{J}} \right)$$

   Given that f_0 describes particles in thermal equilibrium at inverse temperature $\beta = (k_\mathrm{B}T)^{-1}$, show that

$$\mathbf{D}_1 = \mathbf{D}_2 \cdot \mathbf{K}\,,$$

   where $\mathbf{K}$ is a vector function of $\mathbf{J}$ that should be identified.