A15090W1

PUBLIC EXAMINATION

Honour School of Mathematical and Theoretical Physics (MMathPhys)

Master of Science in Mathematical and Theoretical Physics (MScMTP)

Groups and Representations

FRIDAY, 11TH JANUARY 2019, from 9:30 am to 12:30 pm

Answer **three** *out of four questions.*

Start the answer to each questions on a new page.
Calculators are not allowed.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do NOT turn over until told that you may do so.

1.)

(a) Define the terms "normal sub-group", "group homomorphism" and "group representation". Show that the kernel of a group homomorphism $f : G \to \tilde{G}$ is a normal sub-group of G. Also show that for a normal sub-group N of G, the quotient G/N can be given a group structure. [6 marks]

(b) Consider the general linear group $\mathrm{Gl}(\mathbb{C}^n)$ and its sub-group $N = \{\lambda \mathbb{1}_n \,|\, \lambda \in \mathbb{C} \setminus \{0\}\}$. Why is N a normal sub-group of $\mathrm{Gl}(\mathbb{C}^n)$? Denote by $[g]$ the equivalence class in $\mathrm{Gl}(\mathbb{C}^n)/N$ which contains $g \in \mathrm{Gl}(\mathbb{C}^n)$. For a representation $R : G \to \mathrm{Gl}(\mathbb{C}^n)$ of a group G, define a map from G to $\mathrm{Gl}(\mathbb{C}^n)/N$ by $\tilde{R}(g) := [R(g)]$. Show that $\tilde{R}$ is a group homomorphism. [5 marks]

(c) Find all the elements of the group G generated by the two matrices

$$g_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad g_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

In particular, show that G is of order 8 and that it is non-Abelian. [6 marks]

(d) Since the group G from part (c) is defined in terms of matrices it represents itself. Denote this "fundamental representation" of G by $R : G \to \mathrm{Gl}(\mathbb{C}^2)$. From part (b) this representation R leads to a group homomorphism $\tilde{R} : G \to \mathrm{Gl}(\mathbb{C}^2)/N$, where $N = \{\lambda \mathbb{1}_2 \,|\, \lambda \in \mathbb{C} \setminus \{0\}\}$. Work out $\tilde{R}$ explicitly and show that $\mathrm{Im}(\tilde{R}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. [8 marks]

2.) The group G is generated by the two matrices

$$g_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^* \end{pmatrix} , \qquad g_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

where $\alpha = e^{2\pi i/3}$.

(a) Work out the elements of the group G and its conjugacy classes. [7 marks]

(b) Show that the two-dimensional representation R_2 given by the matrices which define the group is irreducible. How many complex, irreducible representations does G have and what are their dimensions? [4 marks]

(c) Work out the complex, irreducible representations of G explicitly and write down the character table of G. [7 marks]

(d) Show that the representation R_2 and its complex conjugate representation R_2^* are equivalent. Find the Clebsch-Gordon decompositions of the tensor products $R_2 \otimes R_2$ and $R_2 \otimes R_2 \otimes R_2$. [7 marks]

3.) Consider the groups $SU(n)$, $U(n)$ and $U(1) \times SU(n) = \{(z, U) \,|\, z \in U(1)\,,\; U \in SU(n)\}$.

(a) Show that the map $f : U(1) \times SU(n) \to U(n)$ defined by $f((z, U)) := zU$ is a group homomorphism. [2 marks]

(b) Use the homomorphism from part (a) to show that

$$U(n) \cong \frac{U(1) \times SU(n)}{\mathbb{Z}_n} \tag{1}$$

and specify the explicit form of the sub-group $\mathbb{Z}_n$. [4 marks]

(c) For the group $SU(6)$, write down highest weight Dynkin labels, Young tableaux and associated tensors for the fundamental representation, the complex conjugate of the fundamental representation, the rank two symmetric and rank two anti-symmetric tensors of the fundamental representation, and the adjoint representation. What are the dimensions of these representations? [8 marks]

(d) For the $SU(6)$ representations R from part (c), define associated representations $\tilde{R}$ of $U(1) \times SU(n)$ by $\tilde{R}((z, U)) := R(U)$, that is, by representing the $U(1)$ factor trivially. In view of the isomorphism in Eq. (1), which of these representations $\tilde{R}$ can be thought of as representations of $U(6)$? [4 marks]

(e) Embed $SU(5)$ into $SU(6)$ via

$$U \to \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}$$

where $U \in SU(5)$. Given this embedding, how do the $SU(6)$ representations from part (c) branch into $SU(5)$ representations? [7 marks]

4.) Consider the group $SO(5)$ with Lie-algebra $so(5)$. The Cartan matrix for $B_2 = so(5)_{\mathbb{C}}$ is

$$A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} .$$

(a) Work out the Lie algebra $so(5)$ by finding the conditions on the matrices in $so(5)$, choose a simple basis for this algebra and find its dimension and rank. [5 marks]

(b) Draw the Dynkin diagram for B_2. Find the weight systems of the B_2 representations with highest weights (in the Dynkin basis) $(0, 1)$, $(1, 0)$ and $(0, 2)$. [8 marks]

(c) Denote by r the B_2 representation with highest weight $(0, 1)$. Which irreducible representations does $r \otimes r$ contain? [6 marks]

(d) Find the representation r explicitly by writing down the representation matrices for the $so(5)$ basis matrices from part (a). (Hint: Think about the spinor representation of $so(5)$.) [6 marks]