Final Honour School of Mathematical and Theoretical Physics Part C and MSc Mathematical and Theoretical Physics

Galactic and Planetary Dynamics

The steps for each part of the miniproject are for your guidance; if you wish to take an alternative route to the desired goal, you are free to do so. But, if you follow the suggested route and find yourself unable to carry out any particular step, you may simply assume it so that you can continue with the miniproject, but should make this assumption clear in your presentation.

Please typeset your solutions using a font size of at least 10 pt for the main body of your report.

In this miniproject you will examine a simple model for how a two-dimensional, collisionless stellar disc responds when it is stirred by an externally imposed potential of the form

$$\Phi_{\rm p}(R,\varphi,t) = \epsilon_{\rm p}\Phi_{\rm a}(R,\varphi-\Omega_{\rm p}t), \qquad (1)$$

in which the amplitude $\epsilon_{\rm p}(t)$ and pattern speed $\Omega_{\rm p}(t)$ of the stirring potential may depend on time t, but its shape $\Phi_{\rm a}(R,\varphi)$ is fixed. Examples of such perturbations include a central bar (see reference [1] below) or spiral arms [2]; we strongly recommend that you read these two papers. Estimates of the pattern speeds $\Omega_{\rm p}$ of the bar and spiral arms of our own Galaxy are given in [3].

We assume that before the disc is stirred it is perfectly axisymmetric with potential $\Phi_0(R)$.

(i) The motion of any star in the unperturbed disc can be decomposed into radial oscillations of frequency κ superimposed upon circular motion of frequency Ω. Explain how to calculate Ω and κ from $\Phi_0(R)$ given the energy E and angular momentum L of the star. Obtain simplified expressions for these frequencies in the special case of a star on an almost circular orbit of radius $R_{\rm c}$.

(ii) Resonant orbits satisfy a commensurability condition,

$$m(\Omega-\Omega_{\rm p}) = l\kappa, \qquad (2)$$

between the star's natural frequencies (Ω,κ) and the stirring frequency $\Omega_{\rm p}$, where l and m are integers. Find the radius corresponding to the (l,m) resonance for stars on almost circular orbits in the potential $\Phi_0(R)=\log(R)$.

(iii) Explain how the motion of a star close to an (l,m) resonance may be approximated by the one-dimensional Hamiltonian

$$H(q,p,t) = \frac{1}{2}Ap^2 - B(t)p + C(t)\cos q, \qquad (3)$$

and, without detailed calculation, outline how the new variables q and p are defined and how to obtain A, $B(t)$ and $C(t)$ from Φ_0 and Φ_p.

(iv) Indicating the method you use to integrate Hamilton's equations, write a computer program that integrates orbits in the Hamiltonian (3). Use your program to follow orbits having initial conditions $p = (2, 3, 4)$ and $q = 0$ for the case $A = B = -C = 1$ and explain how to use these numerically integrated orbits to test your program.

(v) Now consider the case in which the coefficients in the Hamiltonian (3) are

$$A = 1,$$

$$B(t) = \begin{cases} 0, & t < 0, \\ 3\min(1, t/T), & t \geq 0, \end{cases}$$

$$C(t) = \begin{cases} 0, & t < 0, \\ -\sin^2(\pi t/T), & 0 < t < T, \\ 0, & t > T, \end{cases}$$

where T is a constant. At $t \ll 0$ the phase-space density of a sample of test stars is

$$f_0(q, p) = \begin{cases} \frac{1}{4\pi}, & \text{if } -\pi < q < \pi \text{ and } -1 < p < 1 \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Stating clearly the method you use to obtain your results, plot the phase-space density of these stars in the limit $t \gg T$ for the cases $T = 1$, 5, 10, 20 and 100. Identify the key differences among these final distributions and explain their origin.

(vi) Justifying your reasoning, estimate the value of the parameter T that is appropriate for the Galactic disc in the neighbourhood of the sun. How might the model above be tested observationally?

References

1. Chiba R., Friske J., Schönrich R., 2021, MNRAS, 500, 4710
2. Sridhar S., 2019, ApJ, 884, 3
3. Li Z., et al., 2016, ApJ, 824, 13