Final Honour School of Mathematical and Theoretical Physics Part C and MSc Mathematical and Theoretical Physics

Galactic and Planetary Dynamics

The steps for each part of the miniproject are for your guidance; if you wish to take an alternative route to the desired goal, you are free to do so. But, if you follow the suggested route and find yourself unable to carry out any particular step, you may simply assume it so that you can continue with the miniproject, but should make this assumption clear in your presentation.

The numbers in square brackets in the right hand margin indicate the marks that the examiners anticipate assigning to each part.

Please write or print on one side of the paper only.

In this miniproject you will investigate motion in the Hamiltonian

$$\begin{aligned} H(x,y,p_x,p_y) = {} & \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 - (xp_y - yp_x) \\ & - \frac{1}{\sqrt{(x+\mu)^2+y^2}} - \frac{\mu}{\sqrt{(x-1)^2+y^2}}. \end{aligned} \tag{1}$$

It is convenient to rewrite this as $H = H_0 + H_1$, where

$$\begin{aligned} H_0 &= \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 - \frac{1}{\sqrt{x^2+y^2}} - (xp_y - yp_x), \\ H_1 &= \frac{1}{\sqrt{x^2+y^2}} - \frac{1}{\sqrt{(x+\mu)^2+y^2}} - \frac{\mu}{\sqrt{(x-1)^2+y^2}}. \end{aligned} \tag{2}$$

(i) What physical system does the Hamiltonian (1) model? Explain carefully what the coordinates (x, y, p_x, p_y) and the parameter μ represent. [10]

(ii) With reference to H_0 define the following orbital elements: semimajor axis a, eccentricity e, argument of pericentre ω and mean anomaly w. Introducing the longitude $\lambda \equiv \omega + w$, outline briefly the significance of the new coordinates

$$\begin{aligned} (J_\lambda, J_\omega) &= \sqrt{a}\left(1, 1 - \sqrt{1-e^2}\right), \\ (\theta_\lambda, \theta_\omega) &= (\lambda, -\omega), \end{aligned}$$

and show that, in these new coordinates,

$$H_0 = -\frac{1}{2J_\lambda^2} + (J_\omega - J_\lambda).$$

[10]

(iii) Introducing polar coordinates (r, ϕ), defined by $(x, y) = r(\cos\phi, \sin\phi)$, show that

$$\phi = \lambda + 2e \sin w + O(e^2).$$

Use the expansion

$$(1 + a^2 - 2a\cos\phi)^{-s} = \sum_{j=0}^{\infty} b_s^{(j)}(a) \cos j\phi,$$

to obtain an expression for H_1 as a Fourier series in the angles $(\theta_\lambda, \theta_\omega)$ correct to first order in e and μ, stating clearly any further assumptions that you need to make. Leave your answer in terms of the Laplace coefficients $b_s^{(j)}(a)$. [25]

(iv) Now consider the situation in which the semimajor axis a is close to $a_0 \equiv (3/2)^{2/3}$. What would happen if we were to try to use your expansion for H_1 to follow the motion in this case? By making an appropriate sequence of canonical maps, explain why the motion is approximately described by a new Hamiltonian

$$H_s(\theta_s, J_s) = \frac{1}{2} K J_s^2 + \mu e B \cos\theta_s, \tag{3}$$

where $\theta_s = 3\lambda - \omega$, $J_s = J_\lambda - 2J_\omega + \text{constant}$, $K = -18\left(\frac{2}{3}\right)^{\frac{1}{3}}$ and

$$B = -2b_{1/2}^{(2)}(a_0) - \frac{1}{2}a^2 b_{3/2}^{(2)}(a_0) + \frac{1}{4}a\left[b_{3/2}^{(1)}(a_0) + b_{3/2}^{(3)}(a_0)\right].$$

[15]

(v) Construct a numerical integrator to follow orbits in the full Hamiltonian (1). Describe *two* tests can you make to check your implementation. Use your integrator to verify that the approximate Hamiltonian (3) does provide a good description of the motion for at least one orbit having $a \simeq a_0$ for at least value of μ. Assume that $B \simeq -8$. [40]

Hint: Motion under H_0 alone can be integrated exactly by making the substitutions $z = x + \mathrm{i}y$, $p_z = p_x + \mathrm{i}p_y$. To test the integrator, is there a value of μ for which motion under the full Hamiltonian is integrable? When looking for an orbit and a value of μ suitable for verifying H_s you may find it helpful to consider parameters appropriate for bodies within our solar system.