Final Honour School of Mathematical and Theoretical Physics Part C and MSc Mathematical and Theoretical Physics

Galactic and Planetary Dynamics

The steps for each part of the miniproject are for your guidance; if you wish to take an alternative route to the desired goal, you are free to do so. But, if you follow the suggested route and find yourself unable to carry out any particular step, you may simply assume it so that you can continue with the miniproject, but should make this assumption clear in your presentation.

Your miniproject should be 10–15 pages long. Please write or print on one side of the paper only.

Consider an idealised model of a galaxy in which the Hamiltonian for a single star is the usual $H(\mathbf{x},\mathbf{v}) = \frac{1}{2}\mathbf{v}^2 + \Phi(\mathbf{x},t)$, but space is a periodic cube whose sides have length 2π. This model was introduced by Barnes, Goodman & Hut (1986 ApJ 300 112) and analysed further by Weinberg (1993 ApJ 410 543). The periodicity of space means that any function $g(\mathbf{x})$ of the spatial coordinates $\mathbf{x} = (x, y, z)$ can be expanded as the Fourier series

$$g(\mathbf{x}) = \sum_{\mathbf{n}} g_{\mathbf{n}} \mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\mathbf{x}}, \tag{1}$$

in which the sum is over triplets of integers $\mathbf{n} = (n_x, n_y, n_z)$. Note that there is no such periodicity in velocities $\mathbf{v}$!

(i) The simplest possible equilibrium galaxy model in this space has $\Phi(\mathbf{x}, t) = 0$. What is the most general form for the phase space distribution function (hereafter DF), $F(\mathbf{x},\mathbf{v})$, of such a model?

(ii) This equilibrium is perturbed by the introduction of a matter distribution $\epsilon\rho_{\mathrm{e}}(\mathbf{x}, t)$, in response to which the DF changes from $F(\mathbf{x},\mathbf{v})$ to $F(\mathbf{x},\mathbf{v}) + \epsilon f(\mathbf{x},\mathbf{v}, t)$. Stating clearly any assumptions that you make, show that the Fourier modes $f_{\mathbf{n}}$ of the DF response satisfy

$$\frac{\partial f_{\mathbf{n}}}{\partial t} + \mathrm{i}\mathbf{n}\cdot\mathbf{v} f_{\mathbf{n}} + \frac{4\pi G}{\mathbf{n}^2}\mathrm{i}\mathbf{n}\cdot\frac{\partial F}{\partial \mathbf{v}}\left[\rho_{\mathrm{e}\mathbf{n}}(t) + \rho_{\mathbf{n}}(t)\right] = 0, \tag{2}$$

in which $\rho_{\mathbf{n}}(t) \equiv \int \mathrm{d}^3\mathbf{v}\, f_{\mathbf{n}}(\mathbf{v}, t)$. This expression fails for $\mathbf{n} = (0, 0, 0)$. Write down an explicit expression for $f_{(0,0,0)}(\mathbf{v}, t)$ in terms of $f(\mathbf{x},\mathbf{v}, t)$. By examining carefully how you obtained (2), or otherwise, show that $f_{(0,0,0)}(\mathbf{v})$ does not change with time.

(iii) Introducing the Fourier transform $\tilde{f}_{\mathbf{n}}(\mathbf{k}) \equiv \int \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{v}} f_{\mathbf{n}}(\mathbf{v})\mathrm{d}^3\mathbf{v}$, show that $f_{\mathbf{n}}(\mathbf{k}, t)$ satisfies the quasilinear partial differential equation

$$\frac{\partial \tilde{f}_{\mathbf{n}}}{\partial t} - \mathbf{n}\cdot\frac{\partial \tilde{f}_{\mathbf{n}}}{\partial \mathbf{k}} - \frac{4\pi G}{\mathbf{n}^2}\mathbf{n}\cdot\mathbf{k}\left[\rho_{\mathrm{e}\mathbf{n}}(t) + \tilde{f}_{\mathbf{n}}(\mathbf{k} = 0, t)\right]\tilde{F}(\mathbf{k}) = 0. \tag{3}$$

What physical quantity does the $\tilde{f}_{\mathbf{n}}(\mathbf{k}=0,t)$ term in the square brackets represent?

(iv) A galaxy having a Maxwellian DF

$$F(\mathbf{v}) = \frac{\rho_0}{(2\pi)^{3/2}\sigma^3}\mathrm{e}^{-v^2/2\sigma^2} \tag{4}$$

with $\rho_0 = \sigma = 1$ is "stirred" from $t=0$ onwards by the introduction of a point mass M that moves with velocity $\mathbf{V} = (3,3,0)$. That is,

$$\rho_{\mathrm{e}}(\mathbf{x},t) = \begin{cases} 0, & t \le 0, \\ M\delta(\mathbf{x}-\mathbf{V}t), & t > 0. \end{cases} \tag{5}$$

Ignoring the $\tilde{f}_{\mathbf{n}}(\mathbf{k}=0,t)$ term, use (3) to find an explicit expression for $\tilde{f}_{\mathbf{n}}(\mathbf{k},t)$. Hence, or otherwise, plot the galaxy's response densty $\rho(\mathbf{x},t)$ at time $t=1$, explaining the numerical scheme you use to obtain $\rho(\mathbf{x},t)$. If the imposed point mass M were released and allowed to move freely from $t=1$, how would it be affected by this $\rho(\mathbf{x})$?

(v) In reality the neglected $\tilde{f}_{\mathbf{n}}(\mathbf{k}=0,t)$ term can be important. Using the method of characteristics to solve (3), or otherwise, show that

$$\rho_{\mathbf{n}}(t) = 4\pi G \int_0^t (t-t')\left(\rho_{\mathrm{e}\mathbf{n}}(t') + \rho_{\mathbf{n}}(t')\right)\tilde{F}(-\mathbf{n}(t-t'))\,\mathrm{d}t', \tag{6}$$

and, without further numerical calculation, outline *two* methods that could be used to obtain the density response $\rho_{\mathbf{n}}(t)$. Explain how the magnitude of this density response is likely to differ from that obtained using the method of part (iv) above when $F(\mathbf{v})$ is modified by (a) increasing ρ_0 or (b) decreasing σ.