Final Honour School of Mathematical and Theoretical Physics Part C and MSc Mathematical and Theoretical Physics

Galactic and Planetary Dynamics

The steps for this miniproject are for your guidance only; if you wish to take an alternative route to the desired goal, you are free to do so. But, if you follow the suggested route and find yourself unable to carry out any particular step, you may simply assume it so that you can continue with the miniproject, but should make this assumption clear in your presentation.

Please write or print on one side of the paper only.

The purpose of this mini project is to understand the properties of orbits of stars within a massive disc around a black hole. Such orbits can be modelled by the Hamiltonian (in dimensionless units)

$$H_\epsilon(r,\phi,p_r,p_\phi) = \frac{1}{2}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) - \frac{1}{r} + \epsilon\Phi_\star(r,\phi),$$

in which

$$\Phi_\star(r,\phi) = \frac{1}{2}r^2\left(c_0 + c_1\cos\phi + c_2\cos 2\phi\right),$$

and c_0, c_1 and c_2 are constants.

(a) Consider first the Kepler Hamiltonian H_0 obtained by setting $\epsilon = 0$. Define the semimajor axis a, eccentricity e, argument of pericentre ω and mean anomaly w. Starting from the Hamilton–Jacobi equation explain how to construct angle–action coordinates $(\theta_r, \theta_\phi, J_r, J_\phi)$ for H_0. Show that these $(\theta_r, \theta_\phi, J_r, J_\phi)$ can be transformed to a second set of angle-action coordinates $(\theta_a, \theta_b, J_a, J_b)$ in terms of which H_0 becomes a function of J_b only. Obtain the relationship between these $(\theta_a, \theta_b, J_a, J_b)$ and the orbital elements (a, e, ω, w).

(b) Let us average the motion over the "fast" angle θ_b to obtain the averaged Hamiltonian

$$\bar{H}_\epsilon \equiv \oint \frac{\mathrm{d}\theta_b}{2\pi} H = -\frac{1}{2J_b^2} + \epsilon\bar{\Phi}_\star,$$

where

$$\bar{\Phi}_\star \equiv \oint \frac{\mathrm{d}\theta_b}{2\pi}\Phi_\star.$$

By writing

$$\bar{\Phi}_\star = \oint \frac{\mathrm{d}\eta}{2\pi}(1 - e\cos\eta)\Phi_\star,$$

where η is a new variable that you should identify, show that

$$\epsilon\bar{\Phi}_\star = \frac{1}{4}a^2\epsilon\left[(3e^2+2)c_0 - e(e^2+4)c_1\cos\omega + 5e^2c_2\cos 2\omega\right].$$

(c) What do the curves $\bar{\Phi}_\star(\theta_a, J_a)$ = constant represent? Plot contours of $\bar{\Phi}_\star(\theta_a, J_a)$ for the case $(c_0, c_1, c_2) = (1, 0, 0.1)$ at $J_b = 1$. Use this plot to discuss the qualitative properties of orbits supported by H_ϵ for this choice of (c_0, c_1, c_2).

(d) Write a numerical orbit integrator for H_ϵ, justifying carefully your choice of integration scheme and timestep. Setting $\epsilon = 0.1$ and $(c_0, c_1, c_2) = (1, 0, 0.1)$, plot (x, y) traces of orbits launched from $(r, \phi, p_\phi) = (1, 0, 0.8)$, $(1, 0, 0.4)$ and $(1, \pi, 0.4)$, in each case choosing $p_r > 0$ so that $H = -\frac{1}{2}$. Follow each orbit for at least one full precession period.

(e) Construct a (ϕ, p_ϕ) surface of section by following a representative sample of orbits having $H = -\frac{1}{2}$ and leaving consequents in the (ϕ, p_ϕ) plane when each orbit passes through pericentre. Comment on the similarity (or otherwise) to the plot you obtained for part (c).

(f) By considering how the number and nature of the fixed points of $\bar{\Phi}_\star$ depend on (c_0, c_1, c_2), or otherwise, find at least one other orbit family in H_ϵ.

(g) Under what circumstances is such averaging appropriate? Taking $(c_1, c_2, c_3) = (1, 0, 0.1)$, give an example of an orbit in H_ϵ for which the averaged Hamiltonian $\bar{\Phi}_\star(\theta_a, J_a)$ fails to provide a good approximation to the motion.