A15278S1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

COLLISIONLESS PLASMA PHYSICS
TAKE-HOME EXAM

HILARY TERM 2019

TUESDAY, 12TH MARCH 2019, 12noon to THURSDAY 14TH MARCH 2019, 12noon

You should submit answers to all questions. Answer booklets are provided for you to use but you may type your answers if you wish. Typed answers should be printed single-sided and the pages securely fastened together.

You may refer to books and other sources when completing the exam but should not discuss the exam with anyone else.

Do not turn this page until you are told that you may do so

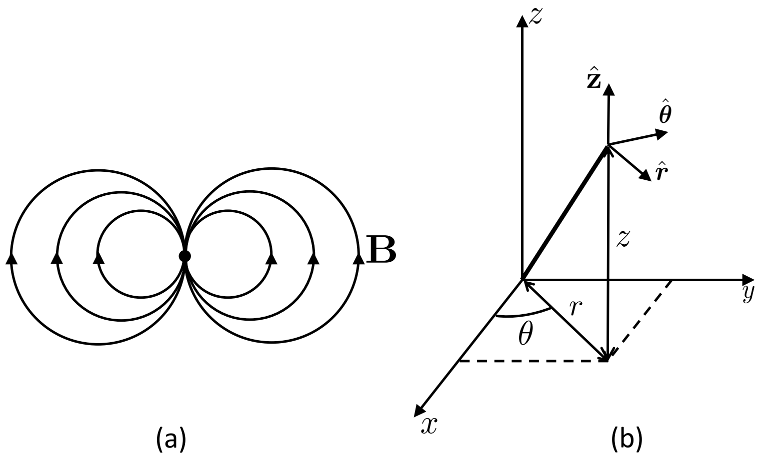

Figure 1: (a) Dipolar magnetic field. (b) Cylindrical coordinates.

1. Consider the dipolar magnetic field shown in figure 1(a). The magnetic field of a dipole is

$$\mathbf{B}(r, z) = B_r\hat{\mathbf{r}} + B_z\hat{\mathbf{z}} = -\frac{1}{r}\frac{\partial \psi}{\partial z}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial \psi}{\partial r}\hat{\mathbf{z}}, \tag{1}$$

where the flux function $\psi(r, z)$ is

$$\psi(r, z) = -\frac{Ar^2}{(r^2 + z^2)^{3/2}}. \tag{2}$$

Here A is a positive constant, $\{r, \theta, z\}$ are the usual cylindrical coordinates, shown in figure 1(b), and $\{\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{z}}\}$ are the usual unit vectors associated to the cylindrical coordinates, also shown in figure 1(b).

Particles of charge Ze and mass m move in the dipolar magnetic field (1). There is an electrostatic electric field $\mathbf{E} = -\nabla\phi$ present. The corresponding electrostatic potential depends only on the flux function $\psi(r, z)$, that is, $\phi(r, z) = \phi(\psi(r, z))$. Assume that the characteristic gyroradius ρ of the particles is small compared to the characteristic size of the system. Assume as well that the force due to the electric field $\mathbf{E}$ is comparable in size to the magnetic force on the particles. Using the lowest-order guiding centre equations, answer the following questions.

(a) [5 marks] Show that particles move on surfaces of constant $\psi(r, z)$ and that their velocity in the θ-direction is of the form $v_\theta = r\Omega_\theta(\psi)$, where the rotation frequency $\Omega_\theta(\psi) = \mathrm{d}\phi/\mathrm{d}\psi$ only depends on ψ.

(b) [10 marks] Show that the quantity

$$U = \frac{1}{2}mv_\parallel^2 + m\mu B - \frac{1}{2}m\Omega_\theta^2(\psi)r^2 \tag{3}$$

is conserved. Here, $v_\parallel$ is the velocity of the particle parallel to the magnetic field, and μ is the magnetic moment of the particle.

(c) [8 marks] Sketch the magnitude of the magnetic field B as a function of r on a surface $\psi(r, z) =$ constant, and use this sketch to briefly describe the particle motion for (i) $\mu = 0$, and (ii) $m\mu B \gg m\Omega_\theta^2 r^2/2$.

(d) [7 marks] Calculate the distribution function of the particles $f(r, z, v_\parallel, \mu)$ everywhere given the distribution function at $z = 0$, which is

$$f(r, z = 0, v_\parallel, \mu) = n_0\left(\frac{m}{2\pi T_0}\right)^{3/2}\exp\left(-\frac{m(v_\parallel^2/2 + \mu B(r, z = 0))}{T_0}\right). \tag{4}$$

The density n_0 and the temperature T_0 at $z = 0$ do not depend on r. Using the distribution function that you have obtained, calculate the density $n = \int f \, \mathrm{d}^3 v$ everywhere. Why does the density have a gradient?

2. The lower-hybrid resonance is used in magnetic confinement devices to drive plasma current. To model the propagation of the wave into the plasma, consider a plasma in a constant magnetic field $B\hat{\mathbf{z}}$ composed of ions with charge Ze and mass m_i and electrons with charge $-e$ and mass m_e. The ion and electron densities $n_s(x)$ depend on x, and they satisfy quasineutrality, $Zn_i(x) = n_e(x)$. The value of the electron plasma frequency $\omega_{pe}(x)$ ranges from zero at the edge of the plasma to a maximum value $\omega_{pe,M}$ that is comparable to the electron gyrofrequency Ω_e. The wavevector of the propagating wave is $\mathbf{k} = k_\perp \hat{\mathbf{x}} + k_\| \hat{\mathbf{z}}$, and its frequency satisfies $\omega \sim \sqrt{\Omega_i \Omega_e}$, where Ω_i is the ion gyrofrequency.

   (a) [5 marks] For densities such that $\omega_{pe} \sim \Omega_e$, show that the elements $\epsilon_\perp$, g and $\epsilon_\|$ of the cold plasma dielectric tensor can be approximated by

   $$\epsilon_\perp \simeq 1 + \frac{\omega_{pi}^2}{\Omega_e \Omega_i} - \frac{\omega_{pi}^2}{\omega^2}, \quad g \simeq -\frac{\omega_{pi}^2}{\omega \Omega_i}, \quad \epsilon_\| \simeq -\frac{\omega_{pe}^2}{\omega^2}, \tag{5}$$

   where ω_{pi} is the ion plasma frequency.

   (b) [3 marks] Using ray tracing, show that $k_\|$ is constant.

   (c) [7 marks] Show that $k_\perp$ is given by

   $$A \left(\frac{k_\perp c}{\omega} \right)^4 + B \left(\frac{k_\perp c}{\omega} \right)^2 + C = 0. \tag{6}$$

   Give the coefficients A, B and C as functions of $k_\|$ and of the components of the dielectric tensor $\epsilon_\perp$, g and $\epsilon_\|$.

   (d) [5 marks] At the lower-hybrid resonance, the perpendicular wavevector $k_\perp$ diverges. Argue that the slow wave is the one that resonates. Determine the density $n_{e,LH}$ at which the resonance happens as a function of ω, and hence show that $\omega < \sqrt{\Omega_i \Omega_e}$.

   [Hint: show first that one of the coefficients A, B or C must vanish for $k_\perp$ to diverge.]

   (e) [8 marks] Show that the fast and slow waves have the same phase velocity at the densities

   $$n_{e\pm} = \frac{\epsilon_0 m_i \omega^2}{Ze^2} \left[\frac{k_\| c}{\sqrt{\Omega_i \Omega_e}} \pm \sqrt{1 + \frac{k_\|^2 c^2}{\Omega_i \Omega_e} - \frac{k_\|^2 c^2}{\omega^2}} \right]^2. \tag{7}$$

   (f) [3 marks] Show that the densities n_{e-} and n_{e+} exist only for $k_\| < k_{\|,M}$, where $k_{\|,M}$ is a function of frequency that you must determine.

   (g) [2 marks] For $k_\| < k_{\|,M}$, show that the wave cannot propagate into regions with electron density such that $n_{e-} < n_e < n_{e+}$.

   (h) [5 marks] For $k_\| < k_{\|,M}$, show that $n_{e-} < n_{e,LH}$ and hence the wave cannot reach the lower-hybrid resonance when launched from the edge of the plasma.

   (i) [2 marks] Determine a minimum $k_\|$ for which the wave can propagate from the edge to the lower-hybrid resonance. Based on this value of $k_\|$, argue that the wave cannot be launched in vacuum.

   (For interest: due to this result, lower-hybrid wave antennae must be located inside the plasma, and they can suffer significant damage for this reason.)

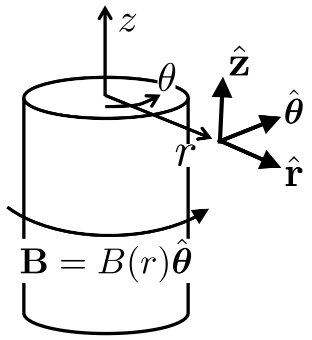

Figure 2: z-pinch and cylindrical coordinates.

3. The z-pinch is a magnetic confinement cylindrical configuration in which the magnetic field closes in azimuthal loops (see Figure 2). The magnitude of the magnetic field depends only on radius, that is, $\mathbf{B} = B(r)\hat{\boldsymbol{\theta}}$. Here we use the usual cylindrical coordinates $\{r, \theta, z\}$ and their associated unit vectors $\{\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{z}}\}$, shown in figure 2.

   The plasma contained in the z-pinch is composed of electrons with charge $-e$ and mass m_e, and one ion species with charge Ze and mass m_i. The distribution functions for both species $(s = i, e)$ are bi-Maxwellians that only depend on the radial position,

$$f_{Bs}(r, v_\parallel, \mu) = n_s(r)\sqrt{\frac{m_s}{2\pi T_{s\parallel}(r)}}\frac{m_s}{2\pi T_{s\perp}(r)}\exp\left(-\frac{m_s v_\parallel^2}{2T_{s\parallel}(r)} - \frac{m_s \mu B(r)}{T_{s\perp}(r)}\right), \tag{8}$$

   where $n_s(r)$, $T_{s\parallel}(r)$ and $T_{s\perp}(r)$ are the density, parallel temperature and perpendicular temperature of species s. The equilibrium electric field is zero, $\mathbf{E} = 0$, and the densities of electrons and ions satisfy quasineutrality, $Zn_i(r) = n_e(r)$. Assume that the ion perpendicular temperature is much smaller than the ion parallel temperature, and that the ion parallel temperature is comparable to the electron parallel and perpendicular temperatures,

$$T_{i\perp} \ll T_{i\parallel} \sim T_{e\parallel} \sim T_{e\perp}. \tag{9}$$

   For stability reasons, z-pinches are run at very low β, where β is the plasma energy divided by the magnetic field energy. We proceed to study the stability of a low-β z-pinch to electric field perturbations of the form $\delta\mathbf{E} = -\nabla\delta\phi$, with

$$\delta\phi(r, \theta, z) = \tilde{\phi}(r)\exp(\mathrm{i}M\theta + \mathrm{i}k_z z - \mathrm{i}\omega t). \tag{10}$$

   We focus on perturbations that satisfy

$$\frac{M}{r}\sqrt{\frac{T_{i\parallel}}{m_i}} \ll \omega \sim \frac{k_z T_{e\parallel}}{eB}\left|\frac{\mathrm{d}}{\mathrm{d}r}\ln n_e\right| \sim \frac{k_z T_{e\parallel}}{eBr} \ll \frac{M}{r}\sqrt{\frac{T_{e\parallel}}{m_e}}. \tag{11}$$

   (a) [3 marks] Before considering the perturbations, show that, for low β, the equilibrium magnetic field satisfies

$$\frac{\mathrm{d}}{\mathrm{d}r}\ln B \simeq -\frac{1}{r}. \tag{12}$$

   (b) [5 marks] Show that the perturbed electron density is given by

$$\tilde{n}_e = \frac{e\tilde{\phi}}{T_{e\parallel}}n_e. \tag{13}$$

(c) [12 marks] Show that the perturbed ion density is given by

$$\tilde{n}_i = \frac{Ze\tilde{\phi}}{T_{i\|}} n_i \Bigg\{ \frac{r}{2L_{T_{i\|}}} - 1 + \frac{1}{2\zeta}\left[\frac{r}{2L_n} + \left(\zeta^2 - \frac{1}{2}\right)\frac{r}{2L_{T_{i\|}}} - \zeta^2 - \frac{1}{2}\right] \\ \times \left[\mathcal{Z}(\zeta) - \mathcal{Z}(-\zeta) + 2\mathrm{i}\sqrt{\pi}\exp(-\zeta^2)\right] \Bigg\}, \tag{14}$$

where $\zeta = \sqrt{\omega/\omega_{di}}$, $\omega_{di} = 2k_z T_{i\|}/eBr$ is the frequency associated to the ∇B and curvature drifts, $L_n = (-\mathrm{d}\ln n_i/\mathrm{d}r)^{-1}$ and $L_{T_{i\|}} = (-\mathrm{d}\ln T_{i\|}/\mathrm{d}r)^{-1}$ are the density and perpendicular temperature characteristic lengths, and $\mathcal{Z}(\zeta)$ is the plasma dispersion function.

[Hint: to obtain this result, decompose $(u^2 - \zeta^2)^{-1}$ into simpler fractions, and be very careful with the difference between the Landau contour of the integral over the resonant denominator $(u + \zeta)^{-1}$ and the Landau contour used in the definition of $\mathcal{Z}(\zeta)$.]

(d) [10 marks] For $1/L_{T_{i\|}}$ sufficiently large, the plasma becomes unstable. To calculate the stability boundary, solve the dispersion relation for a purely real ω. Argue that the mode cannot become unstable when $\omega/\omega_{di} < 0$, and hence that ω/ω_{di} must be positive at the stability boundary. In addition to finding the critical value for $r/L_{T_{i\|}}$, show that

$$\frac{r}{L_n} \leqslant 2 + \frac{T_{i\|}}{ZT_{e\|}} \tag{15}$$

for the plasma to become unstable.