Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

COLLISIONLESS PLASMA PHYSICS
TAKE-HOME EXAM

HILARY TERM 2017

MONDAY, 13 MARCH 2017, 12noon to WEDNESDAY, 15 MARCH 2017, 12noon

You should submit answers to all questions. Answer booklets are provided for you to use but you may type your answers if you wish. Typed answers should be printed single-sided and the pages securely fastened together.

You may refer to books and other sources when completing the exam but should not discuss the exam with anyone else.

1. The hard-core z-pinch is a cylindrical magnetic confinement configuration with azimuthal magnetic field $\mathbf{B} = B(r)\hat{\boldsymbol{\theta}}$ (see Figure 1). The axial current that produces the azimuthal magnetic field runs through a cylindrical conductor of radius r_c (the hard core) at the center of the hard-core z-pinch.

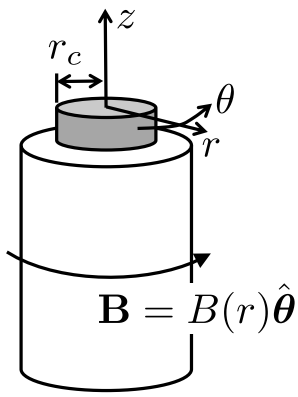

Figure 1: Hard-core z-pinch.

Consider a plasma confined in a hard-core z-pinch composed of one ion species with charge Ze and mass m_i and electrons of charge $-e$ and mass m_e.

(a) [10 marks] Using kinetic MHD, show that one possible steady-state solution is to have zero electric field and gyroaveraged distribution functions $\langle f_s \rangle_\varphi(r, v_\parallel, \mu)$ (where $s = i, e$ refers to ions and electrons, respectively) that only depend on radius r, parallel velocity $v_\parallel$ and magnetic moment μ. Show as well that the distribution functions must have densities n_s, parallel average velocities $u_{s\parallel}$ and perpendicular and parallel pressures, $p_{s\perp}$ and $p_{s\parallel}$, that satisfy quasineutrality, $Zn_i(r) = n_e(r)$, and the condition

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{B^2}{2\mu_0} + P_\perp\right) + \frac{1}{r}\left(\frac{B^2}{\mu_0} + P_\perp - P_\parallel - n_i m_i u_{i\parallel}^2\right) = 0, \tag{1}$$

where $P_\perp = p_{i\perp} + p_{e\perp}$ and $P_\parallel = p_{i\parallel} + p_{e\parallel}$ are the total perpendicular and parallel pressures, and μ_0 is the vacuum permeability.

(b) [5 marks] The plasma is confined to the region $r_c < r < r_p$. Assuming $u_{i\parallel} = 0$, $P_\perp = P_\parallel \equiv P$ and that $B(r) = B_0$ is constant inside the plasma, calculate $P(r)$ for $r_c < r < r_p$ and $B(r)$ for $r > r_p$. Assume that the pressure vanishes at $r = r_p$, $P(r = r_p) = 0$.

2. Consider a hard-core z-pinch for which the magnetic field magnitude starts increasing in time at $t = 0$, that is, $\mathbf{B}(r,t) = B(r,t)\hat{\boldsymbol{\theta}}$, with $\partial \ln B/\partial t \sim v_{ti}/r_c$ and v_{ti} the ion thermal speed.

(a) [5 marks] Show that the axial electric field generated by the change in magnetic field is

$$E_z(r,t) = \int_{r_c}^{r} \frac{\partial B(r',t)}{\partial t}\,\mathrm{d}r'. \tag{2}$$

Assume that the electric field in the radial and azimuthal directions is zero, and that the hard core is a perfect conductor, that is, $E_z(r = r_c) = 0$. Is this electric field high flow or low flow?

(b) [10 marks] Give the guiding center equations of motion for an ion of charge Ze and mass m_i to lowest order in $\rho_{i*} = \rho_i/r_c \ll 1$ during the magnetic field change. Here ρ_i is the ion gyroradius. Prove that the axial angular momentum $L_z = m_i r v_\parallel$ is conserved.

(c) [10 marks] Show that during the change in the magnetic field, the quantity

$$\Psi(r,t) = \int_{r_c}^{r} B(r',t)\,\mathrm{d}r' \tag{3}$$

is a conserved quantity for the particle, that is, $\Psi(r(t),t) =$ constant. Here $r(t)$ is the radial position of the particle as a function of time.

(d) [5 marks] Using the results in (b) and (c), calculate $r(t)$ and $v_\parallel(t)$ for a particle initially at $r(t=0) = r_0$ with parallel velocity $v_\parallel(t=0) = v_{\parallel 0}$ in a magnetic field $B(r,t) = B_0(r_c/r)[1+\tanh(t/t_0)]$. What is the state of the particle at $t \to \infty$?

3. Consider a plasma composed of one ion species of charge Ze and mass m_i, and electrons of charge $-e$ and mass m_e in a strong constant magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ and a zero electric field, $\mathbf{E} = 0$. The gyroaveraged distribution functions for ions and electrons are Maxwellians with average parallel flows $u_{s\parallel}$,

$$\langle f_s \rangle_\varphi(x, v_\parallel, \mu) = f_{Ms}(x, v_\parallel, \mu) \equiv n_s(x) \left(\frac{m_s}{2\pi T_s}\right)^{3/2} \exp\left(-\frac{m_s[(v_\parallel - u_{s\parallel})^2/2 + \mu B]}{T_s}\right). \tag{4}$$

Here $s = i, e$ refers to ions and electrons, respectively. The densities $n_s(x)$ and average flows $u_{s\parallel}(x)$ depend on x, but the temperatures T_s are constant. The densities satisfy quasineutrality, $Zn_i(x) = n_e(x)$, and we assume $T_i \sim T_e$ and $u_{i\parallel} \sim u_{e\parallel} \sim v_{ti}$, where $v_{ti} = \sqrt{2T_i/m_i}$ is the ion thermal speed.

The parameter β is sufficiently small that we can assume that any electric field generated by the plasma is electrostatic and low flow, that is, $\mathbf{E} = -\nabla\phi \sim T/eL_n$, where

$$L_n = -\left(\frac{\mathrm{d}\ln n_e}{\mathrm{d}x}\right)^{-1} \tag{5}$$

is the characteristic length of density variation. Study the stability of this system to perturbations of the form $\delta\mathbf{E} = -\nabla\delta\phi$, where $\delta\phi = \tilde{\phi}(x)\exp(-\mathrm{i}\omega t + \mathrm{i}k_y y + \mathrm{i}k_z z)$ is the perturbed electrostatic potential. The frequency and the wavenumbers of the perturbation satisfy

$$k_z v_{ti} \lesssim \omega \sim \omega_{*e} \ll k_z v_{te}, \tag{6}$$

where $v_{te} = \sqrt{2T_e/m_e}$ is the electron thermal speed and $\omega_{*e} = k_y T_e/eBL_n$ is the drift wave frequency.

(a) [5 marks] Show that the perturbed density of the electrons follows a Maxwell-Boltzmann response, $\tilde{n}_e = (e\tilde{\phi}/T_e)n_e$.

(b) [10 marks] Show that the perturbed ion density is

$$\tilde{n}_i = \left\{\frac{\omega_{*i}}{|k_z|v_{ti}}\left[\mathcal{Z}(\zeta_i) - \frac{2L_n}{v_{ti}}\frac{k_z}{|k_z|}\frac{\mathrm{d}u_{i\parallel}}{\mathrm{d}x}(1+\zeta_i\mathcal{Z}(\zeta_i))\right] - 1 - \zeta_i\mathcal{Z}(\zeta_i)\right\}\frac{Ze\tilde{\phi}}{T_i}n_i, \tag{7}$$

where $\mathcal{Z}$ is the plasma dispersion function, $\zeta_i = (\omega - k_z u_{i\parallel})/|k_z|v_{ti}$ and $\omega_{*i} = -k_y T_i/ZeBL_n$.

(c) [5 marks] Assuming that $(L_n/v_{ti})(\mathrm{d}u_{i\parallel}/\mathrm{d}x) \sim |\zeta_i| \gg 1$ and $\mathrm{Im}(\zeta_i) \geqslant 0$, show that the dispersion relation is

$$(\omega - k_z u_{i\parallel})^2 - \omega_{*e}(\omega - k_z u_{i\parallel}) + \omega_{*e}k_z L_n \frac{\mathrm{d}u_{i\parallel}}{\mathrm{d}x} = 0. \tag{8}$$

Neglect exponentially small terms, that is, terms of order $\exp(-|\zeta_i|^2)$.

(d) [5 marks] Find the values of $\mathrm{d}u_{i\parallel}/\mathrm{d}x$ for which the plasma is unstable and calculate the growth rate.

4. Consider a plasma in a constant magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ composed of one ion species with charge Ze and mass m_i, and electrons of charge $-e$ and mass m_e. The electron and ion densities depend on x, $n_s(x)$. A wave with frequency close to the electron gyrofrequency Ω_e is launched into the plasma to heat it up. To both propagate into the plasma and impart energy to the plasma, the wavevector must have components perpendicular and parallel to the magnetic field. Assume that $\mathbf{k} = k_\perp \hat{\mathbf{x}} + k_\parallel \hat{\mathbf{z}}$.

   (a) [5 marks] Using ray tracing, show that the parallel wavevector $k_\parallel$ is constant along the path of the wave. Since the wave satisfies $\omega \simeq \Omega_e$, argue that launching the wave from vacuum requires $|k_\parallel| c/\Omega_e < 1$.

   (b) [10 marks] Using the cold plasma dispersion relation, show that $k_\perp$ as a function of ω and $k_\parallel$ is given by a quartic,

$$A \left(\frac{k_\perp c}{\omega} \right)^4 + B \left(\frac{k_\perp c}{\omega} \right)^2 + C = 0. \quad (9)$$

   Give the coefficients A, B and C as functions of $k_\parallel$ and of the components of the dielectric tensor $\epsilon_\parallel$, $\epsilon_\perp$ and g.

   (c) [5 marks] Check that the dispersion relation in (b) gives the correct results for vacuum, parallel propagation and perpendicular propagation.

   (d) [10 marks] At some x, if the peak plasma density is sufficiently large, the wave will reflect. Using the quartic in (b), how can you find an equation for the cut-off density without solving for $k_\perp$? There are three different cut-off densities. Calculate them neglecting the ion contribution and assuming that $|\omega - \Omega_e| \ll \Omega_e$ and that the maximum value of ω_{pe} is of the order of Ω_e (note that $\omega_{pe} = 0$ outside of the plasma).