Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

ASTROPHYSICAL GAS DYNAMICS
Trinity Term 2017

WEDNESDAY, 14 JUNE 2017, 12noon to FRIDAY 16 JUNE 2017, 12noon

You should submit answers to questions 1 to 3 and choose one of the essay questions.

Answer booklets are provided for you to use but you may type your answers if you wish. Typed answers should be printed single-sided and the pages securely fastened together.

Please use different booklets for Sections A and B.

You may refer to books and other sources when completing the exam but should not discuss the exam with anyone else.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told you may do so

Section A: Astrophysical Gas Dynamics

1. Gravitational collapse of an isothermal sphere [50 points]

We consider a spherically symmetric cloud made of ideal gas with uniform and constant temperature T.

(a) Write the mass conservation equation using as variables the velocity v and mass $M(r)$ contained within the sphere of radius r. Write the momentum equation using as variables v, $M(r)$ and the mass density ρ. [6]

(b) We look for self–similar solutions under the form:

$$\rho(x,t) = \frac{\alpha(x)}{4\pi}\tilde{\rho}(t), \quad M(x,t) = \tilde{M}(t)m(x), \quad v(x,t) = \tilde{v}(t)u(x),$$

with $x \equiv r/(ct)$, c being the isothermal sound speed. The functions α, m and u are dimensionless. Using dimensional analysis, express $\tilde{\rho}(t)$, $\tilde{M}(t)$ and $\tilde{v}(t)$ in terms of c, t and the gravitational constant G.

Show that α, m, u and x satisfy the relation:

$$m = (x-u)\,x^2\alpha,$$

and the system of differential equations:

$$\begin{aligned} \left[(x-u)^2 - 1\right]\frac{du}{dx} &= \left[(x-u)\,\alpha - \frac{2}{x}\right](x-u)\,, \\ \left[(x-u)^2 - 1\right]\frac{1}{\alpha}\frac{d\alpha}{dx} &= \left[\alpha - \frac{2}{x}(x-u)\right](x-u)\,. \end{aligned}$$

[12]

(c) Show that the hydrostatic solution to these equations is $u = 0$ and $\alpha = 2/x^2$. [4]

We assume that initially the cloud is at hydrostatic equilibrium, that is to say the solutions found in (c) are the initial conditions. At $t = 0$, a perturbation causes the central regions of the cloud to collapse. Gravitational collapse starts at $r = 0$ and progresses outward in the form of an expansion wave. Therefore, hydrostatic equilibrium is maintained beyond some critical radius $r_c(t)$ which increases with time.

(d) Show that the critical radius $r_c(t)$ corresponds to $x = 1$, that is to say the transition between the collapsing and hydrostatic regions of the sphere is at $x = 1$. What is the physical interpretation of this result? Using a series expansion of u and α in the vicinity of $x = 1$, calculate du/dx and $d\alpha/dx$ at $x = 1^-$. [10]

(e) Given $\lim_{x\to 0} m(x) = m_0$ (obtained by numerical integration of the equations), calculate $M(0,t)$. What is the mass accretion rate at $r = 0$? [2]

(f) Calculate the asymptotic behaviour of $u(x)$ and $\alpha(x)$ for $x \to 0$. Interpret the result by considering the behaviour of $v(r)$ near the centre. Sketch α and u as a function of x. [12]

(g) If the cloud is made of hydrogen molecules and has a temperature $T = 10$ K, how long does it take to form a central core with a mass of one solar mass? We give $m_0 = 0.975$. [2]

(h) Which process, not taken into account here, would limit accretion onto the central core? [2]

Section B: Accretion Disc Theory

Please answer **both questions 2 and 3** *and choose* **one of the essay questions**.

2. Consider a stationary accretion disc around a Schwarzschild black hole of mass M. Explain the importance of the *last stable orbit*, $r_{\rm lso}$, the *Eddington limit* and the associated *Eddington accretion rate*. Show that the latter is given by

$$\dot{M}_{\rm edd} = \frac{4\pi c R}{\kappa},$$

where R is the radius (in the case of spherical accretion) and κ the opacity of the gas. [6]

Adopting the Shakura-Sunyaev model for a stationary, optically thick accretion disc (i.e. starting with equations (20)–(24) in the lecture notes), show that, for a disc in which radiation pressure dominates and the opacity is given by pure electron scattering (i.e. $\kappa = 0.034\,{\rm m}^2/{\rm kg}$ for a solar-type plasma), the disc scale height, H, and the radial drift velocity, u, as a function of radius r and accretion rate $\dot{M}$ are given by

$$H = \frac{3\kappa}{8\pi c}\,\dot{M}\,f^4,$$

and

$$\frac{u}{v_{\rm K}} = \frac{1}{r^2}\left(\frac{3\kappa\dot{M}}{8\pi c}\right)^2 f^4\,\alpha,$$

where $v_{\rm K}$ is the local Keplerian velocity and f and α are as defined in the lectures. [10]

Express H and $u/v_{\rm K}$ in terms of $\dot{M}/\dot{M}_{\rm Edd}$, taking R to be $r_{\rm lso}$ in the latter. What happens in the limit $\dot{M} \to \dot{M}_{\rm Edd}$? Comment on the validity of the model in this limit. [4]

3. As was shown in the lectures, in the case of a stationary, optically thick accretion disc radiating as a black body, the effective temperature, $T_{\rm eff}$, as a function of radius, r, is given by

$$\sigma T_{\rm eff}^4 = \frac{3GM\dot{M}}{8\pi r^3}\,f^4, \quad \text{with} \quad f^4 = 1 - \left(\frac{r_0}{r}\right)^{1/2},$$

where M is the central mass, $\dot{M}$ the mass accretion rate and r_0 the inner radius of the disc.

Consider the mass flow in the outer disc, i.e. at a radius $r \gg r_0$ (so that you can set $f \simeq 1$). Show that the energy released between two radii r_1 and r_2 is three times the energy expected to be released in that region from the virial theorem. Explain this suprising result. [5]

Now consider the disc near the inner radius r_0. Explain why the main contribution to the integrated luminosity of the disc comes from the region around r where $\sigma T_{\rm eff}^4 r^2$ is a maximum. Show that this maximum occurs at $r_{\rm max} = 2.25\,r_0$. Explain how this affects the determination of central black-hole masses based on the spectrum of an accretion disc. [5]

Essay Questions

Write a short essay (of the order of 1000 words) on **one** *of the two topics below.*

Essay A: The Magnetorotational Instability

Write an essay on the magnetorotational instability (Balbus-Hawley instability). Specifically, discuss (a) the physical motivation (the viscosity problem), (b) how the instability works, including some simple estimates of its strength (growth rate), (c) applications to accretion discs, and (d) possible limitations of the theory. [20]

Essay B: Super-Eddington Accretion

Write an essay on the possibility of super-Eddington accretion in X-ray binaries in the context of ultraluminous X-ray binaries (ULXs). Specifically, discuss (a) the controversy concerning ULXs, (b) the assumptions behind the classical derivation of the Eddington limit and their limitations, (c) examples of systems accreting above the Eddington limit, (d) some models/ideas that have been proposed to violate it, including a critical assessment of these proposals, and (e) how the possibility of such super-Eddington accretion affects the problem of the growth of supermassive black holes found at the centres of galaxies (you may assume that accretion in these systems can potentially exceed the Eddington limit by a factor of $10 - 100$). [20]