Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

ADVANCED QUANTUM THEORY: PATH INTEGRALS AND MANY-PARTICLE PHYSICS

Trinity Term 2020

Opening time: WEDNESDAY, 10TH JUNE 2020, 09:30 am BST

You should submit answers to both of the two questions.

You have **3 hours** *to complete the paper and upload your answer file.*
You are permitted to use the following material(s):
Calculator
The use of computer algebra packages is **not** *allowed*

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

1. A one-dimensional Ising model with vacancies can be modelled as follows: at lattice sites $i = 0, 1, \ldots, L-1$ "spin" variables σ_i are allowed to take values $\sigma_i \in \{-1, 0, +1\}$, where '0' corresponds to the absence of a spin (a vacancy). The density of vacancies is controlled by a chemical potential μ, while the spins interact via a nearest-neighbor ferromagnetic Ising exchange interaction J. There is *no* external magnetic field. The energy for a spin and vacancy configuration $\{\sigma_i\}$ in this model is therefore given by [25]

$$E[\{\sigma_i\}] = \sum_{i=0}^{L-1} \left[-J\sigma_i\sigma_{i+1} - \mu\delta_{\sigma_i,0}\right], \tag{1}$$

where $\delta_{a,b}$ is the Kronecker delta. You may assume $J, \mu > 0$ unless otherwise stated.

(a) Assuming periodic boundary conditions $\sigma_0 \equiv \sigma_L$, show that the partition function $\mathcal{Z}$ for the model (1) can be expressed as $\mathcal{Z} = \mathrm{Tr}[\hat{T}^L]$ where the transfer matrix $\hat{T}$ takes the form [6]

$$\hat{T} = \begin{pmatrix} a & \sqrt{b} & a^{-1} \\ \sqrt{b} & b & \sqrt{b} \\ a^{-1} & \sqrt{b} & a \end{pmatrix}, \tag{2}$$

and hence determine a and b in terms of J, μ, and $\beta \equiv 1/k_{\mathrm{B}}T$, where T is the temperature and k_{B} is the Boltzmann constant.

(b) Explaining all your reasoning, use your results from part (a) to derive a formula for the free energy per site f in the limit of large L, as a function of β, μ, and J. You may use the fact that the eigenvalues of $\hat{T}$ are [9]

$$\lambda_1 = a - a^{-1}, \quad \lambda_{2,3} = \frac{1}{2}\left(a + a^{-1} + b \pm \sqrt{(a + a^{-1} + b)^2 - 4b(a + a^{-1} - 2)}\right).$$

(c) Assuming that $J \ll \mu$ and working in the regime where $k_{\mathrm{B}}T \gg J$, determine the average density of vacancies, given by [6]

$$n_{\mathrm{av}} = \frac{1}{L}\left\langle \sum_i \delta_{\sigma_i,0} \right\rangle_\beta, \tag{3}$$

where $\langle \ldots \rangle_\beta$ denotes a thermal average. Comment on the limiting behavior when $k_{\mathrm{B}}T \gg \mu$. [*Hint: find an expression for n_{av} in terms of f, and simplify the latter as much as possible in the regime of interest.*]

(d) In this part of the problem, we relax the assumption that $\mu > 0$. Obtain the limiting value of the free energy from part (b) for $\mu \to -\infty$, and use this to compute the entropy per site in the high-temperature limit $k_{\mathrm{B}}T \gg J$. Comment on why these results are physically sensible. [4]

2. A one-dimensional quantum spin model has the Hamiltonian [25]

$$H = -J \sum_{j=0}^{L-1} \left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z \right), \tag{1}$$

where $J > 0$, $0 < \Delta < 1$ and we have a spin-S on each site of the lattice, i.e. $(S_j^x)^2 + (S_j^y)^2 + (S_j^z)^2 = S(S+1)$. We impose periodic boundary conditions, $S_L^\alpha = S_0^\alpha$ for $\alpha = x, y, z$.

(a) Justifying your answer with a calculation, show that for the given values of Δ, in the classical ground state all the spins are aligned parallel to each other and lie in the $S^x S^y$ plane. [*Hint: Consider* (S_j^x, S_j^y, S_j^z) *to be a classical vector of length* S.] [3]

(b) What symmetry is broken by the classical ground state? [2]

In the remainder of this problem, we explore the theory of quantum fluctuations about the classical ground state, that, without loss of generality, we take to be aligned along S^x.

(c) We define a Holstein-Primakoff representation by [4]

$$S_j^x = S - a_j^\dagger a_j, \qquad S_j^+ = S_j^y + iS_j^z = \left(2S - a_j^\dagger a_j\right)^{1/2} a_j, \qquad [a_j, a_\ell^\dagger] = \delta_{j,\ell}$$

[*Note that this convention for the Holstein-Primakoff representation differs from that used in the lectures, due to the different alignment of the ground state.*]

Explain the nature and usefulness of this representation. Identify at least two complications that could generally arise.

(d) Using the Holstein-Primakoff approach, carry out an expansion of H (remembering that $\Delta < 1$) in inverse powers of S. Ignore the constant contribution and drop all terms that grow more slowly than S when S becomes large. Show that the resulting Hamitonian $H_{\rm LSW}$, the linear spin wave (LSW) approximation to H, takes the form [5]

$$H_{\rm LSW} = JS \sum_{j=0}^{L-1} \left[-A(a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j) - B(a_j a_{j+1} + a_j^\dagger a_{j+1}^\dagger) + C a_j^\dagger a_j \right],$$

where A, B, C are positive constants. Determine the values of constants. [*Hint: to get the correct value of* C*, you must use the fact that the* $a_j^\dagger a_j$ *term on site* j *receives contributions from different terms in the sum.*]

(e) By Fourier transforming and using any necessary symmetry properties of the sums over momenta, show that $H_{\rm LSW}$ can be written in the form [3]

$$H_{\rm LSW} = JS \sum_k \left[\{C - 2A\cos k\}\, a^\dagger(k) a(k) - B\cos k \left\{ a^\dagger(k) a^\dagger(-k) + a(-k) a(k) \right\} \right]$$

where $[a(k), a^\dagger(k')] = \delta_{k,k'}$, $[a(k), a(k')] = [a^\dagger(k), a^\dagger(k')] = 0$.

(f) Now, perform a Bogoliubov transformation to write $H_{\rm LSW}$ in the form [8]

$$H_{\rm LSW} = \sum_k \epsilon(k) b^\dagger(k) b(k) + \text{const.},$$

and thereby determine the new creation and annihilation operators $b^\dagger(k), b(k)$ in terms of $a^\dagger(k), a(k)$. Here, 'const.' denotes a constant contribution that you need not determine, and $[b(k), b^\dagger(k')] = \delta_{k,k'}$, $[b(k), b(k')] = [b^\dagger(k), b^\dagger(k')] = 0$. Write down an expression for the dispersion of the low-lying spin waves $\epsilon(k)$. Show that as $k \to 0$ the dispersion is approximately linear, $\epsilon(k) \approx v_s|k|$, thereby determining the "spin-wave speed" v_s. Describe how the dispersion changes as $\Delta \to 1^-$.