A15757W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

ADVANCED QUANTUM THEORY: PATH INTEGRAL AND MANY-PARTICLE PHYSICS

Trinity Term 2018

TUESDAY, 12TH JUNE 2018, 2:30pm to 4:30pm

You should submit answers to both questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. Consider a one dimensional Fermi gas described by the Hamiltonian

$$H = \sum_p \epsilon(p)\ c_p^\dagger c_p + i\Delta \sum_{p>0} \left[c_p^\dagger c_{-p}^\dagger - c_{-p} c_p \right] ,$$

where the c's are fermionic annihilation operators with anti-commutation relations

$$\{c_p, c_k^\dagger\} = \delta_{p,k}.$$

The allowed values for p are

$$p = \frac{2\pi}{L} n \ ,$$

where n are integers.

(a) [5 marks] (a) What is the ground state for $\Delta = 0$ and

$$\epsilon(p) = \alpha(p^4 - \beta p^2 + \mu) \ , \qquad \alpha, \beta, \mu > 0.$$

Derive an expression for the ground state energy as a sum.

(b) [5 marks] (b) Show that

$$\begin{pmatrix} c_p \\ c_{-p}^\dagger \end{pmatrix} = \begin{pmatrix} \cos\theta_p & i\sin\theta_p \\ i\sin\theta_p & \cos\theta_p \end{pmatrix} \begin{pmatrix} \alpha_p \\ \alpha_{-p}^\dagger \end{pmatrix} ,$$

defines a Bogoliubov transformation for fermionic creation/annihilation operators if the angles θ_p fulfil a certain condition. What is this condition?

(c) [8 marks] (c) Consider now the Hamiltonian H with

$$\epsilon(p) = \frac{p^2}{2m} + \mu \ ,$$

where $\mu > 0$. Show by an explicit calculation that H can be written as

$$H = \sum_p E(p) \alpha_p^\dagger \alpha_p + \text{const},$$

where

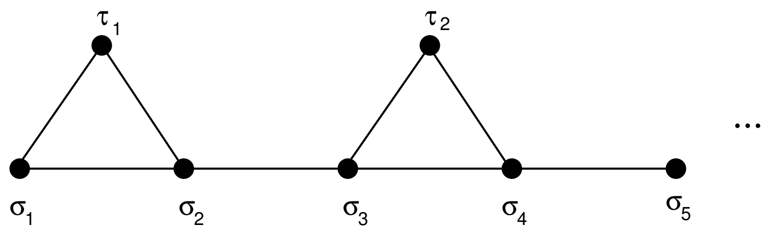
$$E(p) = \begin{cases} \mu & \text{if } p = 0 \\ \sqrt{\epsilon^2(p) + \Delta^2} & \text{else} \end{cases} .$$

(d) [7 marks] (d) Taking into account the zero momentum mode, what is the ground state of H for $\mu > 0$? What is the ground state expectation value of the fermion number operator $\hat{N} = \sum_p c_p^\dagger c_p$? Construct a simple excited state with momentum $k > 0$ and determine the expectation value of $\hat{N}$ in this state. Discuss the physical meaning of your result.

2. (a) [6 marks] (a) Describe the transfer matrix method for calculating the partition function for a system of Ising spins $\sigma_1, \ldots, \sigma_L$ ($\sigma_j = \pm 1$) described by an energy of the form $E = \sum_{j=1}^{L} E_0(\sigma_j, \sigma_{j+1})$ with $E_0(\sigma_j, \sigma_{j+1}) = E_0(\sigma_{j+1}, \sigma_j)$ and periodic boundary conditions $\sigma_{L+1} = \sigma_1$.

(b) [3 marks] (b) Describe how to modify the transfer matrix method for accommodating open boundary conditions, i.e. an energy of the form $E = \sum_{j=1}^{L-1} E_0(\sigma_j, \sigma_{j+1})$ with $E_0(\sigma_j, \sigma_{j+1}) = E_0(\sigma_{j+1}, \sigma_j)$.

(c) [12 marks] (c) Consider an Ising model defined on the lattice shown below (the lattice has $3L/2$ sites, $\sigma_j = \pm 1$, $\tau_k = \pm 1$ and we impose periodic boundary conditions)



The energy is given by

$$E = J \sum_{j=1}^{L} \sigma_j \sigma_{j+1} + J \sum_{j=1}^{L/2} (\sigma_{2j-1} + \sigma_{2j}) \tau_j \ ,$$

where $J > 0$ and we impose periodic boundary conditions $\sigma_{L+1} = \sigma_1$.
Calculate partition function at $T > 0$ by means of the transfer matrix method and show that the free energy per site $f(T)$ in the thermodynamic limit is given by

$$f = -\frac{k_B T}{3} \ln \left(3 + e^{-4\beta J} + e^{-2\beta J} + 3e^{2\beta J} \right) \ .$$

(d) [4 marks] (d) Calculate the entropy per site

$$s = -\frac{\partial f(T)}{\partial T}$$

in the limit of low temperatures. Give a physical interpretation of your result.