A15272W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

ADVANCED QUANTUM FIELD THEORY FOR PARTICLE PHYSICS

Trinity Term 2019

WEDNESDAY, 24TH APRIL 2019, 09:30 am to 12:30 pm

You should submit answers to all three questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. (a) (i) [3 marks] Draw all Feynman diagrams that contribute to the one–loop correction to the gluon self–energy in QCD, ignoring any counterterm contributions.

(ii) [9 marks] Working in $D = 4 - \epsilon$ dimensions, show that the contribution to the gluon self–energy from ghosts is proportional to

$$\Pi^{\mu\nu,ab}_{gh}(k^2) \propto \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{(l+k)^\mu l^\nu}{l^2 (l+k)^2} ,$$

where k is the momentum flowing through the gluon propagator and a, b are the colour indices in the adjoint representation. Working to $O(g^2)$ in the strong coupling g, determine the $O(\frac{1}{\epsilon})$ contribution to this, including all proportionality factors.

(iii) [4 marks] Evaluate the contribution to the gluon self–energy from the diagram(s) featuring the 4–gluon vertex [*Hint: include an artificial gluon mass m_g in the intermediate steps*].

(iv) [4 marks] Using the above results, and without calculating any further Feynman diagrams, evaluate the $O(\frac{1}{\epsilon})$ contribution to the contraction

$$k_\mu \Pi^{\mu\nu,ab}_{3g}(k^2)$$

of the momentum k flowing through the gluon propagator and the contribution to the self–energy due to the diagram(s) featuring 3–gluon vertices, to $O(g^2)$.

[*Selected Feynman rules in Feynman–t' Hooft gauge*:

- *Gluon propagator :* $\frac{-ig_{\mu\nu}}{k^2}\delta^{ab}$.
- *Ghost propagator:* $\frac{i}{p^2}\delta^{ab}$
- *Ghost–antighost-gluon vertex:* $gf^{abc}p^\mu$ *, where p is the momentum flowing along the ghost line pointing away from the vertex, b (c) are associated with the ghost lines flowing towards (away from) the vertex, and a is associated with the gluon.*
- *4-gluon vertex*:

$$-ig^2 \Big[f^{abe} f^{cde} \left(g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho} \right) + f^{ace} f^{bde} \left(g^{\mu\nu} g^{\sigma\rho} - g^{\mu\sigma} g^{\nu\rho} \right)$$
$$+ f^{ade} f^{bce} \left(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} \right) \Big] ,$$

where the vertices μ, ν, ρ, σ are associated with a, b, c, d, respectively. In the above, $a, ..., e$ are the colour indices in the adjoint representation
Feynman's formula to combine denominators:

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \frac{1}{[xA + (1-x)B]^2} ,$$

In $D = 4 - \epsilon$ dimensions,

$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{k^{2a}}{(k^2 + X)^b} = i(-1)^{a-b} \frac{1}{(4\pi)^{D/2}} \frac{1}{(-X)^{b-a-\frac{D}{2}}} \frac{\Gamma\left(a + \frac{D}{2}\right)\Gamma\left(b - a - \frac{D}{2}\right)}{\Gamma(b)\Gamma\left(\frac{D}{2}\right)} ,$$

where $\Gamma(\epsilon) = \frac{1}{\epsilon} + O(\epsilon^0)$.]

(b) [5 marks] The beta function for a generic coupling α can be written as

$$\beta(\alpha) = \frac{\partial \alpha}{\partial \ln \mu} = b_0 \alpha^2 \ ,$$

to $O(\alpha^2)$, where μ is the renomalization scale and b_0 is a constant. Derive an expression for the coupling at scale μ_f in terms of the coupling at a scale μ_i, and discuss the different regimes which exist depending on b_0, providing examples from the gauge couplings of the Standard Model.

2. (a) Consider the QED Lagrangian

$$\mathcal{L}_{\text{QED}} = i\bar{\psi}\not{D}\psi - m\bar{\psi}\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 ,$$

where $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative.

(i) [4 marks] Calculate the change in the Lagrangian $\delta\mathcal{L}_{\text{QED}}$ under the infinitesimal local gauge transformations

$$\begin{aligned} \delta A_\mu &= \partial_\mu \alpha , \\ \delta\psi &= -ie\alpha\psi . \end{aligned}$$

(ii) [4 marks] Now consider the Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{QED}} + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi ,$$

where ϕ is a real scalar field. Show that the generalised transformation

$$\begin{aligned} \delta A_\mu &= \epsilon\partial_\mu\phi , \\ \delta\psi &= -ie\epsilon\phi\psi , \\ \delta\phi &= -\tfrac{\epsilon}{\xi}\partial_\mu A^\mu , \end{aligned}$$

where ϵ is an infinitesimal parameter, leaves the action invariant.

(b) (i) [5 marks] Show that

$$\text{Tr}(\not{a}\gamma^\nu\not{b}\gamma_\mu\not{c}\gamma_\nu\not{d}\gamma^\mu) = -32(a\cdot c)(b\cdot d) ,$$

for arbitrary 4–vectors a, b, c, d, in $D = 4$ dimensions.

(ii) [6 marks] Consider the case of electron–muon scattering in QED, $e^-\mu^- \to e^-\mu^-$ in the high–energy limit, i.e. neglecting fermion masses. Draw the contributing Feynman diagram(s). Averaging over initial–state spins and summing over final–state spins, show that the squared matrix element $\langle|\mathcal{M}|^2\rangle$ is proportional to

$$\langle|\mathcal{M}|^2\rangle \propto \frac{s^2+u^2}{t^2} ,$$

where s, t, u are the Mandelstam variables, and calculate the constant of proportionality.

(iii) [6 marks] Now consider the case of electron–electron scattering in QED, $e^-e^- \to e^-e^-$, again in the high–energy limit. Draw the contributing Feynman diagram(s). Averaging over initial–state spins and summing over final–state spins, calculate the squared matrix element.

[*Recall that the gamma matrices obey:* $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\cdot\mathbb{1}$, *where* $\mathbb{1}$ *is a unit matrix in spinor space.*
You may also assume without derivation: $\text{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$.
The photon propagator in the Feynman–t' Hooft gauge: $\frac{-ig_{\mu\nu}}{k^2}$.
The photon–lepton vertex: $-ie\gamma^\mu$.]

3. (a) [7 marks] Consider the case of a spontaneously broken abelian gauge symmetry. The terms in the Lagrangian relevant to the massive gauge boson are

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}M^2 A^\mu A_\mu \ ,$$

in the unitary gauge, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and M is the gauge boson mass. Using the path integral formalism, compute the gauge boson propagator.

(b) Consider the following Lagrangian

$$\mathcal{L} = \mathrm{Tr}\left(\partial^\mu \Phi^\dagger \partial_\mu \Phi\right) - V\left(\Phi^\dagger \Phi\right) \ ,$$

where Φ is an $N \times N$ matrix of complex scalar fields, $\Phi_{ij}(x)$, $i, j = 1, \cdots, N$.

(i) [4 marks] First take

$$V\left(\Phi^\dagger \Phi\right) = m^2 \mathrm{Tr}\left(\Phi^\dagger \Phi\right) \ .$$

Expanding out the trace to write this Lagrangian in terms of N^2 complex fields, show that it obeys a $SO(2N^2)$ symmetry. What would the corresponding symmetry be for a Hermitian ($\Phi^\dagger = \Phi$) matrix Φ?

(ii) [2 marks] In what follows, we take instead that

$$V\left(\Phi^\dagger \Phi\right) = \frac{\alpha}{2}\mathrm{Tr}\left(\Phi^\dagger \Phi \Phi^\dagger \Phi\right) + \frac{\beta}{2}\left(\mathrm{Tr}\left(\Phi^\dagger \Phi\right)\right)^2 + m^2 \mathrm{Tr}\left(\Phi^\dagger \Phi\right) \ ,$$

where here and in what follows we do not assume that Φ is Hermitian. Show that the Lagrangian is invariant under the global symmetry $G = SU(N)_L \otimes SU(N)_R \otimes U(1)$, acting as

$$\Phi(x) \to e^{i\theta} U_L \Phi(x) U_R^\dagger \ , \qquad U_L, U_R \in SU(N) \ .$$

(iii) [4 marks] For the case of $\alpha, \beta > 0$ and $m^2 < 0$, without loss of generality, the vacuum expectation value (vev) of the potential may be written as

$$\langle \Phi \rangle = C \cdot \mathbb{1}_{N\times N} \ ,$$

where

$$C^2 = -m^2/(\alpha + N\beta) \ .$$

Verify that this breaks the original symmetry of the Lagrangian and show that the remaining symmetry that preserves this vev corresponds to

$$\Phi(x) \to U \Phi(x) U^\dagger \ , \qquad U \in SU(N) \ ,$$

i.e. to $SU(N)$. Hence determine the number of broken generators of the original symmetry.

(iv) [8 marks] Consider the convenient expansion about the vev

$$\Phi(x) = C \cdot \mathbb{1}_{N\times N} + \frac{\phi_1(x) + i\phi_2(x)}{\sqrt{2}}$$

where $\phi_{1,2}$ are Hermitian $N \times N$ matrices. Expanding out the relevant terms in the Lagrangian, show that ϕ_2 corresponds to N^2 massless modes. In light of Goldstone's theorem and the above results, what does this imply for the field content corresponding to ϕ_1?