A15272W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

ADVANCED QUANTUM FIELD THEORY FOR PARTICLE PHYSICS

Trinity Term 2018

WEDNESDAY, 18TH APRIL 2018, 2.30pm to 5.30pm

You should submit answers to all three questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. (a) The Lagrangian for scalar electrodynamics has the form

$$\mathcal{L} = (D_\mu\phi)^* D^\mu\phi - m^2\phi^*\phi - \frac{\lambda}{4}(\phi\phi^*)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} ,$$

where $D_\mu = \partial_\mu + ieA_\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

(i) [3 marks] Write down the renormalized Lagrangian, including counterterms.

(ii) [4 marks] Consider the case that $\lambda = 0$. Draw all Feynman diagrams that in general contribute to the one–loop correction to the 4–point scalar–scalar–photon–photon vertex.

(iii) [8 marks] Setting all external momenta to zero, and working in the Lorenz gauge, show that the one–loop correction to this 4–point vertex is proportional to

$$e^4 \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{g^{\mu\rho}P_{\rho\sigma}(l)g^{\sigma\nu}}{l^2(l^2 - m^2)} + (\mu \leftrightarrow \nu) ,$$

where $P_{\mu\nu}(l) = g_{\mu\nu} - l_\mu l_\nu / l^2$. Then, using dimensional regularization in $D = 4 - \epsilon$ dimensions, determine the renormalization factor, Z_4, for the 4–point vertex, to $O(\frac{1}{\epsilon})$.

[*Feynman rules in the Lorenz gauge:*
- *For each internal scalar,* $\frac{i}{k^2 - m^2}$.
- *For each internal photon,* $-i\frac{P_{\mu\nu}(k)}{k^2}$.
- *For each scalar–scalar–photon vertex,* $-ie(k + k')_\mu$, *where* k *and* k' *are the incoming and outgoing scalar 4–momenta.*
- *For each scalar–scalar–photon–photon vertex,* $2ie^2 g_{\mu\nu}$.

Feynman's formula to combine denominators:

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \frac{1}{(xA + (1-x)B)^2} , \tag{1}$$

In $D = 4 - \epsilon$ *dimensions*

$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{k^{2a}}{(k^2 + \Delta)^b} = i(-1)^{a-b} \frac{1}{(4\pi)^{D/2}} \frac{1}{(-\Delta)^{b-a-\frac{D}{2}}} \frac{\Gamma\left(a + \frac{D}{2}\right)\Gamma\left(b - a - \frac{D}{2}\right)}{\Gamma(b)\Gamma\left(\frac{D}{2}\right)} ,$$

where $\Gamma(\epsilon) = \frac{1}{\epsilon} + O(\epsilon^0)$.]

(b) Consider the quark–antiquark pair production process via gluon fusion in QCD. Specifically, focus on $g(k_1)g(k_2) \to u(p_1)\overline{u}(p_2)$. Neglect quark masses throughout.

(i) [7 marks] Draw the contributing tree–level Feynman diagrams. Writing the amplitude in the form $\epsilon_1^\mu(k_1)\epsilon_2^\nu(k_2)\mathcal{M}_{\mu\nu}$, where $\epsilon_{1,2}$ are the gluon polarization vectors, consider the replacement $\epsilon_{1,2} \to k_{1,2}$. Show that the corresponding amplitude is of the form

$$k_1^\mu k_2^\nu \mathcal{M}_{\mu\nu} \propto \left([T^a, T^b] - if^{abc}T^c\right),$$

and determine the proportional factor in terms of Dirac spinors (it is not necessary to perform any explicit contraction).

(ii) [3 marks] What does this imply for $k_1^\mu k_2^\nu \mathcal{M}_{\mu\nu}$? How would this expression be different for QED? Briefly discuss the wider importance of this result.

[*Recall that* $\not{a}\not{b} + \not{b}\not{a} = 2(a \cdot b)\mathbb{1}$.
Selected Feynman rules in Feynman–'t Hooft gauge:

- *Gluon propagator* : $\frac{-ig_{\mu\nu}}{k^2}\delta^{ab}$.
- *3-gluon vertex:* $gf^{abc}((p_1 - p_2)^\gamma g^{\alpha\beta} + (p_2 - p_3)^\alpha g^{\beta\gamma} + (p_3 - p_1)^\beta g^{\gamma\alpha})$, *for incoming momenta* p_1, p_2, p_3 *associated with vertices* α, β, γ *and colour indices* a, b, c, *respectively.*
- *Massless quark propagator:* $i\frac{\not{p}}{p^2}\delta_{ij}$.
- $q\bar{q}g$ *vertex:* $igT^a_{ij}\gamma^\mu$, *where the colour indices* i (j) *are associated with the fermion line pointing away from (towards) the vertex.*

Here i, j *and* a, b *are the colour indices in the fundamental and adjoint representations, respectively.*]

2. (a) Consider the theory of a Yukawa interaction between a real scalar field $\phi(x)$ and a spinor field $\psi(x)$. The interaction Lagrangian is given by

$$\mathcal{L}_{\rm int}(\phi,\overline{\psi},\psi) = g\phi(x)\overline{\psi}(x)\psi(x)\ ,$$

where g is a real coupling constant.

(i) [3 marks] Show that the generating functional can be written as

$$Z[J,\overline{\eta},\eta] = \exp\left[i\int \mathrm{d}^4x\,\mathcal{L}_{\rm int}\left(\frac{1}{i}\frac{\delta}{\delta J(x)}, i\frac{\delta}{\delta\eta(x)}, \frac{1}{i}\frac{\delta}{\delta\overline{\eta}(x)}\right)\right] Z_0[J,\overline{\eta},\eta]\ , \qquad (2)$$

where Z_0 is the generating functional for the free theory.

(ii) [9 marks] We have

$$Z_0[J,\overline{\eta},\eta] = Z_0[0,0,0]\exp\left(\frac{i}{2}\int \mathrm{d}^4x\mathrm{d}^4y J(x)\Delta(x-y)J(y)\right)$$
$$\exp\left(i\int \mathrm{d}^4w\mathrm{d}^4z\overline{\eta}(w)S_F(w-z)\eta(z)\right)\ ,$$

where $\Delta(x-y)$ and $S_F(w-z)$ are the scalar and fermion propagators, respectively. Consider in (2) the action of the $O(g^2)$ term in the expansion of $e^{i\int \mathrm{d}^4x\mathcal{L}_{\rm int}}$ on Z_0. Show that the contribution this gives to the connected scalar two–point correlation function is

$$\langle 0|T\phi(x_a)\phi(y_a)|0\rangle_{O(g^2)} = g^2\int \mathrm{d}^4x\mathrm{d}^4y\Delta(x_a-x)\Delta(y_a-y)\mathrm{Tr}\left[S(x-y)S(y-x)\right]$$

where the trace is over the spinor indices.

(iii) [2 marks] The above expression corresponds to the connected single fermion loop correction to the scalar two–point correlation function. How would it change if the fields $\overline{\psi}$, ψ were real scalars? [No explicit calculation is required].

(b) (i) [8 marks] Determine the form of the ghost Lagrangrian for the case of a non–abelian gauge theory with the choice of gauge fixing function

$$f(A^a) = n^\mu A^a_\mu - \sigma^a(x)\ ,$$

for arbitrary 4–vector n, which is appropriate for the axial gauge. By performing a path integral over σ, weighted by the function $\exp[-\frac{i}{2\xi}\int \mathrm{d}^4x\sigma^a\sigma^a]$, determine the corresponding gauge fixing Lagrangian term.

(ii) [3 marks] The light cone gauge is a special case of the axial gauge, with $\xi=0$ and $n^2=0$. In this gauge, the gluon propagator is given by

$$i\tilde{\Delta}^{\mu\nu}_{ab}(p) = \frac{i}{p^2+i\epsilon}\left(g^{\mu\nu} - \frac{n^\mu p^\nu + n^\nu p^\mu}{(n\cdot p)}\right)\delta^{ab}\ .$$

Using the result from part (i), show that the contribution from ghost fields vanishes in this gauge.

3. (a) Consider the following Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2\right) + \frac{1}{2}\mu^2(\phi_1^2+\phi_2^2) - \frac{1}{4}\lambda(\phi_1^2+\phi_2^2)^2 + i\overline{\psi}\partial\!\!\!/\psi - g\overline{\psi}(\mathbb{1}\phi_1 + i\gamma_5\phi_2)\psi \ ,$$

where $\phi_{1,2}$ are real scalar fields and ψ is a Dirac fermion, while the parameters $\mu^2, \lambda > 0$ and $\mathbb{1}$ is a unit matrix in spinor space.

(i) [2 marks] Show that

$$\exp(i\alpha\gamma_5) = \mathbb{1}\cos\alpha + i\gamma_5\sin\alpha \ .$$

(ii) [5 marks] We require that this Lagrangian be invariant under the global symmetry

$$\psi \to \exp\left(-i\frac{1}{2}\alpha\gamma_5\right)\psi \ ,$$

where α is a real parameter. Show that this implies the scalar doublet $\phi = (\phi_1, \phi_2)$ must obey a global $SO(2)$ symmetry with rotation angle α. [You may find the result from part (i) useful].

(iii) [2 marks] Verify that the entire Lagrangian is indeed invariant under this combined symmetry of the ϕ and ψ fields. Would a mass term for the Dirac field preserve this symmetry?

(iv) [5 marks] Show that the ground state, determined from the minimum of the potential $V(\phi_1, \phi_2)$, breaks the above symmetry spontaneously. Expanding around a conveniently chosen vacuum–expectation value v, show that the fermion acquires a mass, and determine its value in terms of the coupling g and v.

(v) [3 marks] Determine the masses of the remaining scalar bosons in the theory, and any new interaction vertices that have been introduced by this symmetry breaking. [*The γ_5 matrix obeys $\gamma_5^\dagger = \gamma_5$, $\gamma_5^2 = 1$ and $\{\gamma_5, \gamma^\mu\} = 0$.*]

(b) The Lagrangian describing a spontaneously broken abelian gauge theory has the form

$$\mathcal{L} = (D^\mu\phi)^* D_\mu\phi - V(\phi) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \mathcal{L}_{gf} \ ,$$

where in the general R_ξ gauge

$$\begin{aligned} D_\mu\phi &= \frac{1}{\sqrt{2}}\left[(\partial_\mu h + gbA_\mu) + i(\partial_\mu b - g(v+h)A_\mu)\right] \ , \\ \mathcal{L}_{gf} &= -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 + gvA_\mu\partial^\mu b - \frac{1}{2}\xi g^2 v^2 b^2 \ , \end{aligned}$$

for real scalar fields b and h.

(i) [6 marks] Determine the mass of the gauge boson and compute the gauge boson propagator in this gauge.

(ii) [2 marks] Identify the form of the propagator for the b field. Determine the behaviour of this field in the $\xi \to \infty$ limit.