Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

ADVANCED FLUID DYNAMICS

Trinity Term 2019

TUESDAY, 23RD APRIL 2019, 9:30am to 11:30am

You should submit answers to both of the two questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. **Magnetohydrodynamics.**

(a) [4 marks] State the equations of ideal, incompressible magnetohydrodynamics ("iMHD") and explain under what physical and mathematical assumptions they are valid.

(b) [5 marks] Show that these equations can be rewritten as the following closed set describing the evolution of the velocity field $\mathbf{u}$ and the Maxwell tensor $M_{ij} = B_i B_j$, where $\mathbf{B}$ is the magnetic field (in velocity units):

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial r_j} = -\frac{\partial p}{\partial r_i} + \frac{\partial M_{ij}}{\partial r_j}, \tag{1}$$

$$\frac{\partial M_{ij}}{\partial t} + u_n \frac{\partial M_{ij}}{\partial r_n} = M_{nj} \frac{\partial u_i}{\partial r_n} + M_{in} \frac{\partial u_j}{\partial r_n}. \tag{2}$$

Here t is time, $\mathbf{r}$ is position, and summation over repeated indices is implied. How is the pressure p determined?

(c) [2 marks] Imagine that there is no mean magnetic field, and that the iMHD medium is static and filled with chaotically tangled magnetic fields that are constant in time. Denote their Maxwell tensor $M_{ij}^{(0)}$. What equation must $M_{ij}^{(0)}$ satisfy in order for such an equilibrium to exist? What is the physical meaning of this equation?

(d) [2 marks] Assume that these fields have a characteristic scale that is no larger than ℓ and are statistically isotropic, so that, if we introduce an average (denoted by angle brackets) over scales of order ℓ, then

$$\langle M_{ij}^{(0)} \rangle = v_{\mathrm{A}}^2 \delta_{ij}, \tag{3}$$

where v_{A} is a constant. Find this constant in terms of the mean square magnetic field $\langle B^2 \rangle$.

(e) [9 marks] Consider infinitesimal perturbations δu_i and δM_{ij} around this equilibrium and assume that they vary in space on scales much longer than ℓ, viz.,

$$\langle u_i \rangle = 0 + \delta u_i \ll v_{\mathrm{A}}, \quad \langle M_{ij} \rangle = \langle M_{ij}^{(0)} \rangle + \delta M_{ij}, \quad \delta M_{ij} \ll v_A^2. \tag{4}$$

Ignore any possible perturbations of u_i and M_{ij} on scales ℓ or smaller. Show that δu_i and δM_{ij} will be propagating waves, derive their dispersion relation and also the relationship between δM_{ij} and the fluid-displacement vector associated with δu_i. These are called *magnetoelastic waves*. Comment on their physical nature, their similarity with or difference from Alfvén waves.

(f) [3 marks] The ideal iMHD equations written in the form of (1) and (2) are mathematically similar to the equations describing certain kinds of polymer-laden fluids. What property of the magnetic fields is *not* shared by polymer strands in a fluid—and what property of the latter is not shared by the former? If equation (2) were extended to include dissipative effects, how would these differences manifest themselves, physically and mathematically?

2. **Complex Fluids.**

This question concerns Stokes flow around spheres of radius a moving in an unbounded volume of incompressible fluid with viscosity μ.

(a) [10 marks] Show that the flow

$$p(\mathbf{x}) = \frac{3\mu a}{2}\frac{\mathbf{U}\cdot\mathbf{x}}{r^3}, \quad \mathbf{u}(\mathbf{x}) = \mathbf{U}\left(\frac{3a}{4r} + \frac{a^3}{4r^3}\right) + (\mathbf{U}\cdot\mathbf{x})\,\mathbf{x}\,\frac{3}{4}\left(\frac{a}{r^3} - \frac{a^3}{r^5}\right),$$

with $r = |\mathbf{x}|$ satisfies the boundary condition(s) required for the flow around a sphere that is translating with velocity $\mathbf{U}$, and whose centre is instantaneously located at $\mathbf{x} = 0$.

Show that the drag force on the sphere can be computed by integrating a component of the stress $\boldsymbol{\sigma}$ over a large sphere centred at $\mathbf{x} = 0$, and hence show that the drag force is $\mathbf{F} = -6\pi\mu a\mathbf{U}$.

(b) [4 marks] Suppose the centre of the sphere is now at $\mathbf{r}_1$, and the sphere is subjected to an external force $\mathbf{F}_1$. Show that the flow far from the sphere is given at leading order by

$$\mathbf{u}(\mathbf{x}) = \frac{1}{8\pi\mu\, r}\left(\mathbf{I} + \hat{\mathbf{r}}\,\hat{\mathbf{r}}\right)\cdot\mathbf{F}_1,$$

where $\mathbf{r} = \mathbf{x} - \mathbf{r}_1$, $r = |\mathbf{r}|$, $\hat{\mathbf{r}} = \mathbf{r}/r$ is a unit vector, and $\mathbf{I}$ is the identity matrix.

(c) [11 marks] Consider two spheres of radius a with centres at $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ joined by a Hookean spring with spring constant H. The spheres are separated by a distance much larger than a. The surrounding fluid is flowing with velocity $\mathbf{u}(\mathbf{r}) = \mathbf{L}\,\mathbf{r}$ for some constant matrix $\mathbf{L}$, plus disturbances due to the spheres.

By writing the fluid velocity evaluated at the centres of the two spheres as $\mathbf{u}(\mathbf{r}_i, t) = \mathbf{L}\,\mathbf{r}_i + \mathbf{u}'_i$ for $i \in \{1, 2\}$, where $\mathbf{u}'_1$ is the perturbation to the flow created by sphere 2 evaluated at the centre of sphere 1, and vice versa, or otherwise, show that the centre $\frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2)$ moves with the velocity of the undisturbed background flow, and that the separation $\mathbf{R} = \mathbf{r}_1 - \mathbf{r}_2$ evolves according to

$$\dot{\mathbf{R}} = \mathbf{L}\,\mathbf{R} - \frac{2H}{\zeta}\left(\mathbf{I} - \frac{\zeta}{8\pi\mu R}\left(\mathbf{I} + \hat{\mathbf{R}}\hat{\mathbf{R}}\right)\right)\cdot\mathbf{R},$$

where $R = |\mathbf{R}|$ and $\hat{\mathbf{R}} = \mathbf{R}/R$. Determine the constant ζ.

Verify that the additional interaction between the two spheres due to their finite radii is proportional to a/R.