A15090W1

# PUBLIC EXAMINATION

**Honour School of Mathematical and Theoretical Physics (MMathPhys)**

**Master of Science in Mathematical and Theoretical Physics (MScMTP)**

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**Groups and Representations**

**FRIDAY, 13th JANUARY 2017, 9:30am to 12:30pm**

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*Answer* **three** *out of four questions.*

*Start the answer to each questions on a new page.*

*The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.*

**Do NOT turn over until told that you may do so.**

**1.)** Let $G$ be a group.

a) Provide a set of necessary and sufficient conditions for a subset $H$ of $G$ to be a sub-group. Define the term "normal sub-group". Show that, for a normal sub-group $H \subset G$, the quotient $G/H$ is a group. [5]

b) From now on focus on the group $G = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ with the group multiplication being the addition modulo 4. Write down all the complex irreducible representations and the character table of this group and check explicitly that the characters are ortho-normal. [6]

c) Show that $H = \{0, 2\}$ is a normal subgroup of $G$ and write down the complex irreducible representations of $H$. For each complex irreducible representation of $G$ from part b), find the $H$ representation it branches to under the restriction of $G$ to $H$. [6]

d) Given the matrix

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

show that $R : G \to \mathrm{Gl}(\mathbb{C}^4)$ defined by $R(g) := M^g$ is a representation of $G$ and determine which irreducible representations of $G$ it contains. Under the restriction of $G$ to $H$, which irreducible $H$ representations does $R$ contain? [8]

**2.)** Define $\alpha = \exp(2\pi i/5)$ and the two matrices

$$\tau = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} , \qquad \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

a) Show that $\tau$ and $\sigma$ generate a group $G$ of order 10 with elements

$$G = \{\tau^k \,|\, k = 0, \ldots, 4\} \cup \{\sigma\tau^k \,|\, k = 0, \ldots, 4\} .$$

Further, show that $G$ has four conjugacy classes given by $C_1 = \{\mathbb{1}_2\}$, $C_\tau = \{\tau, \tau^4\}$, $C_{\tau^2} = \{\tau^2, \tau^3\}$ and $C_\sigma = \{\sigma\tau^k \,|\, k = 0, \ldots, 4\}$. Since $G$ is given in terms of $2 \times 2$ matrices it trivially defines a two-dimensional complex representation $R$. Show that $R$ is irreducible. [6]

b) How many complex, irreducible representations does $G$ have and what are their dimensions? Find the character table of $G$. [8]

c) Find the character of the tensor representation $R \otimes R$ and find its irreducible representation content. [8]

d) Show that the complex conjugate representation $R^*$ is equivalent to $R$. [3]

**3.)** Consider the group $SU(6)$ of $6 \times 6$ unitary matrices with determinant one.

a) Find the Lie algebra $su(6)$ and the Cartan sub-algebra of $SU(6)$. What is the dimension of this Lie algebra and what is the rank of $SU(6)$? [4]

b) For the $SU(6)$ representations with highest Dynkin weights $(1\,0\,0\,0\,0)$, $(0\,0\,0\,0\,1)$ and $(0\,1\,0\,0\,0)$ write down the associated Young tableaux, the associated tensors in index notation and the dimensions. [5]

c) Construct all possible $SU(6)$ singlets which are cubic in the tensors found in part b). [4]

d) Consider the sub-group $SU(5) \times U(1)$ of $SU(6)$, where the $SU(5)$ factor is embedded in the standard way as

$$SU(5) \ni U_5 \to \begin{pmatrix} U_5 & 0 \\ 0 & 1 \end{pmatrix} \in SU(6) \; .$$

Find the branching of the $SU(6)$ representations in part b) under this $SU(5) \times U(1)$ sub-group. [8]

e) Write the singlets found in part c) in terms of the $SU(5) \times U(1)$ representations in d). [4]

**4.)** The Cartan matrix $A(G_2)$ and the metric $G(G_2)$ of the exceptional Lie algebra $G_2$ are given by

$$A(G_2) = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \; , \qquad G(G_2) = \frac{1}{3} \begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix} \; .$$

a) Construct the weight systems of the $G_2$ representations with highest Dynkin weights $(1\,0)$ and $(0\,1)$. What are the dimensions of these representations (assuming the $(0\,0)$ weight in the representation $(1\,0)$ has degeneracy 2 and all other weights are non-degenerate)? [8]

b) What are the values of quadratic Casimir for the representations in a)? [5]

c) The projection matrix for the sub-algebra $A_2 \subset G_2$ is given by

$$P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \; .$$

Find the $A_2$ weights contained in the branching under $A_2$ of the $G_2$ representations from part a). [6]

d) From the results in part c), identify the $SU(3)$ representations which are contained in the $G_2$ representations from part a). [6]