Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

QUANTUM MATTER: SUPERCONDUCTORS, SUPERFLUIDS, AND FERMI LIQUIDS

Trinity Term 2017

SATURDAY, 3rd JUNE 2017, 2:30pm to 4:30pm

You should submit answers to both of the two questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. In this question we consider an interacting spinless bose superfluid in three dimensions where the interactions are repulsive and short ranged. In the two-fluid model, there is a normal-fluid fractional mass density given by ρ_n and a super-fluid fractional mass density ρ_s. Suppose the total pressure and total density are fixed, and there is no net flow of fluid, $\mathbf{j}_{total} = 0$,

   (a) [2 marks] Given these conditions, write an equation giving the relationship between the normal-fluid velocity and the super-fluid velocity.

   (b) [2 marks] Under these conditions, a temperature gradient causes a velocity of the super fluid fraction given by

   $$\frac{d\mathbf{v}_s}{dt} = K\nabla T \tag{1}$$

   where K is some constant and T is temperature. Give a qualtitative argument to determine the sign of K. Your answer should be a short paragraph or less.

   (c) [4 marks] Given what you know about the low energy excitations of a superfluid, show that at low temperature the heat capacity per unit volume $\tilde{c}_v$ is of the form

   $$\tilde{c}_v = AT^p \tag{2}$$

   for some constant A and exponent p. Determine the exponent p (you do not need to determine A).

   (d) [8 marks] Using the result of part (a,b,c), determine the second sound velocity in terms of the above defined constant K, the temperature T and the mass densities ρ_n, and ρ_s. Hint: You will need to use an equation of current conservation for entropy density.

Let us define a superfluid order parameter $\psi(r,t) = \langle \Psi(t)|\hat{\psi}(r)|\Psi(t)\rangle$ where $\hat{\psi}$ is a boson destruction operator and $\Psi(t)$ is the time dependent superfluid wavefunction (we can assume this wavefunction Ψ does not have fixed particle number, i.e, is a coherent state, so the order parameter is nonzero). We will write ψ in the form $\psi = \sqrt{n_s}\, e^{i\theta}$.

   (e) [1 mark] Derive an expression for the superfluid velocity in terms of θ.

A superfluid analog of the AC Josephson effect is given by

$$-i\hbar\frac{d\psi(r,t)}{dt} = -\mu(r,t)\,\psi(r,t) \tag{3}$$

where μ is the chemical potential.

   (f) [2 marks] Assuming that the superfluid density is completely constant, derive a relationship between the acceleration of the superfluid and the gradient of μ.

   (g) [3 marks] Using the Gibbs-Duheim relationship $Nd\mu = -SdT + VdP$, and still assuming constant P and constant density and very low temperature, find the value of the constant K in Eq. 1, in terms of the constant A in Eq. 2, the temperature, and the total mass density ρ.

   (h) [3 marks] Given your results and what you know about superfluids, how does the second sound velocity scale with temperature as T goes to zero?

2. Consider a one dimensional gas of interacting spin 1/2 fermions with Hamiltonian

$$H = \sum_i \frac{p_i^2}{2m} + \frac{Ud^2}{2} \sum_{i \neq j} \frac{1}{d^2 + (x_i - x_j)^2}$$

where $U \geqslant 0$. Assume periodic boundary conditions in a system of length L which is very large. The fermions have total number density $n = N/L$.

(a) [3 marks] Explain what a Hartree-Fock approximation is in the context of interacting fermi systems. Your answer should be a short paragraph or less.

(b) [2 marks] Rewrite the Hamiltonian in second quantized notation using creation and annihilation operators which have anticommutation relations $\{\hat{\psi}_\sigma(x), \hat{\psi}^\dagger_{\sigma'}(x')\} = \delta_{\sigma,\sigma'}\delta(x - x')$ where σ is the spin index taking the values $\uparrow$ and $\downarrow$.

(c) [3 marks] Rewrite the Hamiltonian in second quantized notation using k-space creation and annihilation operators having anticommutation relations $\{c_{k,\sigma}, c^\dagger_{q,\sigma'}\} = \delta_{\sigma,\sigma'}\delta_{kq}$ where the wavevector is quantized as $k = 2\pi n/L$ with integer n.

(d) [4 marks] For small $U > 0$, calculate the Hartree contribution to total interaction energy of the system. (The hint below may be useful.)

(e) [10 marks] Again assuming small $U > 0$ calculate the Fock contribution to the total interaction energy. (The hint below may be useful.)

(f) [3 marks] Using your Hartree-Fock results, and using thermodyanmic relationships or otherwise, give a Hartree-Fock approximation for the inverse compressibility $d\mu/dn$ of the gas in the low density limit (μ is chemical potential).

Hint: You may find the following integral useful

$$\int dx \frac{e^{ikx}}{d^2 + x^2} = \frac{\pi}{d} e^{-|k|d}$$

for $d > 0$.