# Final Honour School of Mathematical and Theoretical Physics Part C and MSc Mathematical and Theoretical Physics

## Galactic and Planetary Dynamics

*The steps for each part of the miniproject are for your guidance; if you wish to take an alternative route to the desired goal, you are free to do so. But, if you follow the suggested route and find yourself unable to carry out any particular step, you may simply assume it so that you can continue with the miniproject, but should make this assumption clear in your presentation.*

*Please write or print on one side of the paper only.*

The purpose of this miniproject is to demonstrate the use of Lie transforms in constructing *superconvergent* expansions in perturbation theory. The first few steps have been covered in the lecture course, but you should nevertheless provide complete answers in your report. You might find it helpful to do some further reading before tackling the later steps. Possible references include:


- Binney J., Tremaine S., 2008, *Galactic dynamics*
- Cary J.R., 1981, Physics Reports, 79, 129–159
- Chirikov B.V., 1979, Physics Reports, 52, 263–379
- Lichtenberg A.J., Lieberman M.A., 1992, *Regular and Chaotic Dynamics*
- Morbidelli A., 2001, *Celestial mechanics*

**Part 1: a perturbed simple harmonic oscillator**

(a) Outline the construction of angle-action variables $(\theta, J)$ for the simple harmonic oscillator Hamiltonian,

$$H_0(x, p) = \frac{1}{2}p^2 + \frac{1}{2}\omega_0^2 x^2,$$

and show that $H_0(J) = \omega_0 J$.

(b) Now consider the pendulum Hamiltonian,

$$H(\varphi, p) = \frac{1}{2}p^2 - \omega_0^2 \cos\varphi.$$

Plot contours of constant $H$ in the $(\varphi, p)$ phase plane, indicating librating and circulating orbits and the separatrix. Obtain an expression for the oscillation frequency about the lower equilibrium point $\varphi = 0$ as a power series in $E \equiv H + \omega_0^2$, including terms up to $O(E^3)$.

(c) When expressed in terms of angle–action variables for the simple harmonic oscillator, show that the pendulum Hamiltonian becomes

$$H(\theta, J) = H_0(J) + H_1(\theta, J),$$

in which $H_0$ is the simple harmonic oscillator Hamiltonian and $H_1(\theta, J)$ is a perturbation for which you should obtain an expression. Verify that the angle-averaged perturbation is

$$\begin{aligned}\bar{H}_1(J) &\equiv \frac{1}{2\pi}\int_0^{2\pi} H_1(\theta, J)\,\mathrm{d}\theta \\ &= \omega_0^2\left[-\frac{1}{16}\left(\frac{J}{\omega_0}\right)^2 + \frac{1}{288}\left(\frac{J}{\omega_0}\right)^3 - \frac{1}{9216}\left(\frac{J}{\omega_0}\right)^4 + \cdots\right].\end{aligned}$$

**Part 2: perturbation theory**

Now let us construct first- and second-order approximations to the angle–action coordinates for the pendulum. In the following we use $(\theta^0, J^0)$ for the angle-action coordinates of the simple harmonic oscillator and $(\theta^1, J^1)$ for our first-order approximation to the angle-action coordinates for the pendulum. The pendulum hamiltonian is then one function, $H^0(\theta^0, J^0)$, of the angle-action coordinates for the simple harmonic oscillator, and another function, $H^1(\theta^1, J^1)$, of our approximate angle-action coordinates for the pendulum. We also introduce the Lie derivative,

$$L_A f \equiv [f, A],$$

of the function $f$ along the flow generated by the "Hamiltonian" $A$.

(a) By introducing factors of $\epsilon$ at appropriate places and then considering limiting values of $\epsilon$, explain why we may assume that old and new coordinates are related via

$$\begin{pmatrix} \theta^0 \\ J^0 \end{pmatrix} = \exp[\epsilon L_{\chi_1}] \begin{pmatrix} \theta^1 \\ J^1 \end{pmatrix},$$

with

$$H^1(\theta^1, J^1) = \mathrm{e}^{\epsilon \mathcal{L}_{\chi_1}} H^0(\theta^1, J^1),$$

for some function $\chi_1(\theta^1, J^1)$. Hence show that

$$H^1 = H_0 + \epsilon H_1 + \epsilon[H_0, \chi_1] + \epsilon^2[H_1, \chi_1] + \frac{1}{2}\epsilon^2[[H_0, \chi_1], \chi_1] + O(\epsilon^3),$$

indicating clearly the arguments of the functions on the right-hand side.

(b) For the pendulum Hamiltonian find a choice of $\chi_1$ that eliminates the angle dependence of the $(J/\omega)^2$ term in $H^1$.

(c) This idea can be generalized to higher order by considering further transformations of the form

$$\begin{pmatrix} \theta^{r-1} \\ J^{r-1} \end{pmatrix} = \exp[\epsilon L_{\chi_r}] \begin{pmatrix} \theta^r \\ J^r \end{pmatrix}, \quad H^r(\theta^r, J^r) = \mathrm{e}^{\epsilon L_{\chi_r}} H^{r-1}(\theta^r, J^r),$$

in which improved estimates, $(\theta^r, J^r)$, of the angle-action coordinates are constructed from the previous ones, $(\theta^{r-1}, J^{r-1})$. Construct a function $\chi_2$ that eliminates the angle dependence from the $(J/\omega)^3$ term in $H^2(\theta^2, J^2)$.

(d) Ignoring angle-dependent terms and explaining the method you use, overlay contours of constant $H^1$ (from part (b) above) and of constant $H^2$ (from part (c)) in the $(\varphi, p)$ phase plane plotted in Part 1. How well does the oscillation frequency in each case match the power-series expression for the true pendulum Hamiltonian?

### Part 3: superconvergent perturbation theory

Finally, consider the general case in which we have a Hamiltonian of the form

$$H(\boldsymbol{\theta}, \mathbf{J}) = H_0(\mathbf{J}) + \sum_{n=1}^{\infty} \epsilon^n h_n(\boldsymbol{\theta}, \mathbf{J}).$$

Let $T(\epsilon)$ be a mapping to new angle–action coordinates $(\boldsymbol{\theta}', \mathbf{J}')$ defined implicitly by

$$\begin{pmatrix} \boldsymbol{\theta} \\ \mathbf{J} \end{pmatrix} = T(\epsilon) \begin{pmatrix} \boldsymbol{\theta}' \\ \mathbf{J}' \end{pmatrix}.$$

(a) Explain why the Hamiltonian expressed in terms of these new coordinates,

$$H'(\boldsymbol{\theta}', \mathbf{J}') = H_0'(\mathbf{J}') + \sum_{n=1}^{\infty} \epsilon^n h_n'(\boldsymbol{\theta}', \mathbf{J}'),$$

is given by $H' = TH$.

(b) Now expand $T$ in powers of $\epsilon$ in two different ways:

$$T = \sum_{n=0}^{\infty} \epsilon^n T_n = \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} \epsilon^n L_n\right],$$

with $L_n \bullet \equiv [\bullet, \chi_n]$ and $T_0 = 1$. By differentiating $T$ once with respect to $\epsilon$ and equating powers of $\epsilon$, or otherwise, show that the $T_n$ for $n > 0$ obey the recurrence relation

$$nT_n = \sum_{m=0}^{n-1} L_{n-m} T_m.$$

(c) By differentiating the relation $H' = TH$ with respect to $\epsilon$, show that

$$[\chi_n, H_0] = n(h_n' - h_n) + \sum_{m=1}^{n-1} (L_{n-m} h_m' + m T_{n-m} h_m).$$

(d) Suppose that the functions $h_1$,...,$h_{m-1}$ are independent of angle $\boldsymbol{\theta}$. Explain why, by making an appropriate choice of $\chi_m$, ..., $\chi_{2m}$ it is possible to transform to new angle-action coordinates $(\boldsymbol{\theta}', \mathbf{J}')$ in terms of which the Hamiltonian becomes

$$H'(\boldsymbol{\theta}', \mathbf{J}') = H_0(\mathbf{J}') + \sum_{n=1}^{2m-1} \epsilon^n h'_n(\mathbf{J}') + \sum_{n=2m}^{\infty} \epsilon^n h'_n(\boldsymbol{\theta}', \mathbf{J}').$$

Outline the significance of such transformations for systems having two or more degrees of freedom.