A15809W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

Radiative Processes and High Energy Astrophysics

TRINITY TERM 2024

Monday, 10 June, 2:30 pm – 4:00pm

Answer two questions.
Start the answer to each question in a fresh book.
The use of approved calculators is permitted.
A list of physical constants and conversion factors accompanies this paper.
The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. (a) [5 marks] A supernova remnant expands outwards, driving a shock wave into the surrounding interstellar medium. A population of cold electrons interacts with the shock. Describe the first order Fermi acceleration process that can occur in this scenario, specifically the process by which the electrons gain energy and momentum. Assume the shock front obeys the strong shock conditions. Draw a diagram to illustrate the process and derive the average energy gained per crossing of the shock front.

   (b) [6 marks] Give an expression for the energy of an electron after N crossings of the shock front. Provide some reasoning that the probability that an electron will undergo a further collision may be given by $P = 1 - u/c$, where u is the shock velocity and c the speed of light. Describe why the observed spectral energy distribution may therefore resemble a power law, and provide estimates for the spectral index.

   (c) [4 marks] The supernova explosion led to the creation of a heavy neutron star (1.8 solar masses, 12 km radius), which now exists in a binary system with a main sequence star, in a tight orbit. Give a qualitative picture of how the neutron star may be accreting material from the companion star. How may it end up hitting the neutron star considering that the neutron star has a strong dipolar magnetic field?

   (d) [3 marks] Assume there is a shock front forming above a magnetic pole of the neutron star due to in-falling material. Write down the equations for the conservation of mass, momentum, and energy flow per unit area for material moving across the shock front, assuming the temperature in the upstream region of the shock is very small compared to the downstream temperature.

   (e) [4 marks] Assuming an ideal gas, and using the strong jump conditions for velocity and density, use energy conservation to express the downstream temperature as

$$T_d = 3\mu v_u^2/16k_B,$$

   where μ is the mean particle mass of the accreting material, k_B is the Boltzmann constant, and v the velocity of the particles. The subscripts d and u denote downstream and upstream respectively.

   (f) [3 marks] If the in-falling material is ionized hydrogen, estimate the particle energy. Assume the upstream velocity approaches terminal velocity.

2. (a) [4 marks] A potential model for the density profile of stars is:

$$\rho(r) = \rho_c \frac{\sin ar}{ar}$$

for $r \leqslant R$, and zero at larger distances, where R is the radius of the star, and $a \equiv \pi/R$. Sketch this profile as a function of R, and derive an expression for ρ_c for a star of mass M and radius R.

(b) [5 marks] Let us consider a simple model for the gas content of a star, consisting of a neutral plasma of free electrons, protons (i.e. fully-ionised hydrogen), and photons, and let us consider only Thomson scattering between photons and electrons. Using the density profile in (a), show that the mean free path for Thomson scattering within the star depends on the distance from its center as

$$\ell(r) = L \frac{ar}{\sin ar}$$

and find expressions for L and L/R. What is the value of L and L/R for the Sun in this model?

(c) [5 marks] At what distance from the surface of the star, ΔR_{esc}, can photons effectively propagate freely? Calculate ΔR_{esc} and $\Delta R_{\text{esc}}/R$ for the Sun in this model.

(d) [6 marks] Calculate the pressure profile $P(r)$ and the temperature profile $T(r)$ of this star assuming hydrostatic equilibrium.

Hint 1: when solving the hydrostatic equilibrium equation, you will need a boundary condition. What should the pressure be at the surface of the star?

Hint 2: the following integral may be useful

$$\int_y^\pi dx \frac{\sin x[\sin x - x\cos x]}{x^3} = \frac{1}{2}\left(\frac{\sin y}{y}\right)^2 .$$

(e) [5 marks] Describe the shape of the emergent spectrum for this star. If this star was the Sun, what should the peak frequency of this spectrum be, roughly? Is this a reasonable model for the Sun?

3. The bremsstrahlung emissivity is

$$j_\nu = j_0\, Z^2 \left(\frac{n_i}{10^{-3}\,\mathrm{cm}^{-3}}\right) \left(\frac{n_e}{10^{-3}\,\mathrm{cm}^{-3}}\right) \sqrt{\frac{T_0}{T}}\, g(\nu, T)\, e^{-h\nu/(k_B T)},$$

where n_i is the number density of ions, n_e is the number density of electrons, T is the gas temperature, and

$$j_0 = 4.885 \times 10^{-51}\,\mathrm{W\,m^{-3}\,Hz^{-1}\,sr^{-1}}, \quad T_0 = 19.77 \times 10^8\,\mathrm{K}.$$

(a) [2 marks] Explain why $j_\nu \propto n_i\, n_e$, and why $j_\nu \propto Z^2$. What is $g(\nu, T)$?

(b) [3 marks] Describe the shape of this spectrum for $\nu \ll k_B T/h$ and $\nu > k_B T/h$, and explain the reason for this behaviour in each case.

(c) [5 marks] Consider a hot plasma composed of fully-ionised hydrogen and helium, with a hydrogen mass fraction of x_H. Assume from now on that $g(\nu, T) = 1$. Find an expression for the bremsstrahlung emissivity of this plasma in terms of n_e, x_H, and T.

(d) [5 marks] The emission from clusters of galaxies is often quantified in terms of the number of photons emitted per interval of time t and photon energy ε:

$$L_\varepsilon^X \equiv \frac{dN}{d\varepsilon\, dt}.$$

Consider a simple model: a cluster is a sphere of radius R with homogeneous density and temperature. Use the expression you found in (c) for j_ν to show that

$$L_\varepsilon^X = \frac{4\pi}{h\varepsilon} \frac{4\pi R^3}{3} j_0 \frac{2}{1 + x_H} \left(\frac{n_e}{10^{-3}\,\mathrm{cm}^{-3}}\right)^2 \sqrt{\frac{T_0}{T}} e^{-\varepsilon/(k_B T)}.$$

(e) [4 marks] The thermal Sunyaev-Zel'dovich (SZ) effect from a given cluster of galaxies is quantified in terms of the Compton-y distortion, given by

$$y = \frac{\sigma_T}{m_e c^2} \int dx\, n_e\, k_B T,$$

where the integral is along the line of sight, and $\sigma_T/m_e c^2 = 0.004\,\mathrm{keV^{-1}\,Mpc^{-1}\,cm^3}$ (in units that may be useful later). Explain the mechanism behind SZ emission from clusters, and show that, for our cluster,

$$y = \frac{2\sigma_T}{m_e c^2} R\, n_e\, k_B T.$$

(f) [6 marks] We have measured the X-ray luminosity of a given cluster of galaxies for photons of energy $\varepsilon_1 = 0.2\,\mathrm{keV}$ and $\varepsilon_2 = 5\,\mathrm{keV}$, obtaining:

$$L_{\varepsilon_1}^X = 2.88 \times 10^{54}\,\mathrm{keV^{-1}\,s^{-1}}, \quad L_{\varepsilon_2}^X = 4.40 \times 10^{50}\,\mathrm{keV^{-1}\,s^{-1}}.$$

We have also made use of SZ observations to determine the Compton-y distortion of this cluster to be $y = 1.38 \times 10^{-5}$. Use these three measurements to determine the temperature T (in K), the electron density n_e (in cm^{-3}), and the radius R (in Mpc) of the cluster. You may assume a hydrogen mass fraction $x_H = 0.76$.