A15088W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

QUANTUM FIELD THEORY

Hilary Term 2023

WEDNESDAY, 11th JANUARY 2023, 09:30 am to 12:30 pm

You should submit answers to all three questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. The Lagrangian for a real scalar field ϕ *in 2 dimensions* (one space plus time) is given by

$$\mathcal{L} = \frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}(\partial_x\phi)^2 - \frac{1}{2}m^2\phi^2 - V(\phi).$$

   (a) [5 marks] In the case $V(\phi) = \frac{g_4}{4!}\phi^4$, give the momentum space Feynman rules and explain *briefly* their origin.

   (b) [4 marks] Draw the Feynman diagram for the one loop correction to the two point function and evaluate it.

   (c) [7 marks] In a renormalization scheme where m is the pole mass, write down the definition of the mass counter-term δ_m, and evaluate it to one loop. Explain why there is no $\delta_Z\partial_\mu\phi\partial^\mu\phi$ counter-term, and why a ϕ^4 counter-term is not required at one loop.

   (d) [6 marks] Show that the superficial degree of divergence of a diagram with V vertices is given by $\omega = 2(1 - V)$. Hence show that in fact no further counter-term contributions are generated at *any* order in perturbation theory.

   (e) [3 marks] How would your results change if $V(\phi) = \frac{g_4}{4!}\phi^4 + \frac{g_6}{6!}\phi^6$?

$$\left[\text{You may assume that } \int^\Lambda \frac{d^2p}{(2\pi)^2}\frac{1}{p^2 - m^2 + i\epsilon} = \frac{-i}{4\pi}\log\left(\frac{\Lambda^2}{m^2} + 1\right)\right]$$

2. The Lagrangian density for a system in four space-time dimensions consisting of two complex scalar fields Φ, ϕ of masses M, m is given by

$$\mathcal{L} = \partial_\mu\Phi^\dagger\partial^\mu\Phi + \partial_\mu\phi^\dagger\partial^\mu\phi - M^2\Phi^\dagger\Phi - m^2\phi^\dagger\phi - \frac{g}{2}(\phi^2\Phi^\dagger + c.c.).$$

   (a) [5 marks] What is the internal symmetry of $\mathcal{L}$? Find the corresponding conserved current and explain its physical significance.

   (b) [4 marks] Write down the Feynman rules, taking care to show the flow of quantum numbers.

   (c) [9 marks] Draw the tree-level Feynman graphs for:

      i) scattering of two ϕ particles;

      ii) scattering of a ϕ particle and its antiparticle.

   Write down the matrix elements for these two processes. Assuming that $M \gg m$ find the ratio of the total cross-sections in the limit of low momentum scattering (*i.e.* all external momenta satisfy $|\mathbf{p}| \ll m$). Why is this calculation incorrect if $s \approx M^2$?

   (d) [7 marks] Draw the lowest order Feynman graphs for:
   i) scattering of two Φ particles;
   ii) scattering of a Φ particle and its antiparticle.
   Find approximate expressions for these matrix elements that are valid when $|\mathbf{p}| \ll M \ll m$, and give their leading dependence on g and m. [It is not necessary to do a detailed calculation for this part.]

3. Define the matrices

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^1 = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^2 = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

and the tensor $\eta^{\mu\nu} = \mathrm{diag}(1,-1,-1)$, where $\mu,\nu = 0,1,2$.

The action for a massive two component fermion in three dimensions (one time and two space) is given by

$$S = \int d^3x\, \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

where $\overline{\psi} \equiv \psi^\dagger\gamma^0$.

(a) [3 marks] Show that the matrices $\gamma^0, \gamma^1, \gamma^2$ satisfy the relationship

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}.$$

(b) [4 marks] Explain carefully why the spatial inversion operation P in two space dimensions is $(x,y) \to (-x,y)$, whereas in three space dimensions it is $(x,y,z) \to (-x,-y,-z)$.

(c) [6 marks] Under P the field ψ obeys the transformation law $\psi(t,x,y) \to \psi'(t,x,y) = \gamma^1\psi(t,-x,y)$. Show that the kinetic term of the action S is invariant under P, but that the mass term changes sign.

(d) [6 marks] Under the time inversion operation T the field ψ obeys the transformation law $\psi(t,x,y) \to \psi'(t,x,y) = \gamma^2\psi(-t,x,y)$, and $i\gamma^\mu\partial_\mu \to (i\gamma^\mu\partial_\mu)^*$, where $*$ denotes complex conjugate. Find the action of T on the action S and hence show that S is invariant under TP.

(e) [6 marks] The action of the C operation on ψ is $\psi(t,x,y) \to \psi'(t,x,y) = \Gamma\psi(t,x,y)^*$. Find Γ such that S is invariant under C.