MMATHPHYS Mini-Project

Non-equilibrium Statistical Physics
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1. A Master equation for the probability distribution $\mathcal{P}_n(t)$ describing a discrete process characterized by non-negative integers n can be converted into a differential equation for the probability generating function $G(z,t)$, defined as

$$G(z,t) = \sum_n z^n \mathcal{P}_n(t).$$

Let us consider the birth-death process

$$A \xrightarrow{\lambda} A + A, \qquad A \xrightarrow{\mu} 0,$$

subject to the initial condition of $n = n_0$ at $t = 0$. The aim of this mini-project is to derive the exact solution for the above process, using this method.

(a) [5 marks] Show that the process has a mean population of $\bar{n}(t) = n_0 e^{(\lambda-\mu)t}$.

(b) [10 marks] Derive the differential equation that governs the time evolution of the generating function of the above process, denoted as $G_{n_0}(z,t)$ henceforth.

(c) [10 marks] State the boundary conditions and the initial conditions that are to be satisfied by $G_{n_0}(z,t)$.

(d) [20 marks] Derive the exact solution for $G_{n_0}(z,t)$ subject to the above conditions, and express it in terms of $\bar{n}(t)$, μ, λ, and n_0.

(e) [15 marks] Derive the extinction probability $\mathcal{P}_0(t)$ as a function of time and discuss its limiting behaviour.

(f) [15 marks] Calculate the first four cumulants of the distribution from the exact solution of $G_{n_0}(z,t)$, using a cumulant expansion, and express them in terms of t, μ, λ, and n_0.

(g) [10 marks] Find the long-time asymptotic limiting behaviour for the k-th cumulant, corresponding to the two regimes where $\lambda > \mu$ and $\lambda < \mu$.

(h) [15 marks] Examine the special limit in which $\lambda = \mu$ in the exact solution $G_{n_0}(z,t)$, and derive the long-time limiting behaviour of the k-th cumulant in this case.