

MMATHPHYS Mini-Project

Non-equilibrium Statistical Physics
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Checked by:

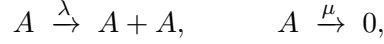
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1. A Master equation for the probability distribution $\mathcal{P}_n(t)$ describing a discrete process characterized by non-negative integers n can be converted into a differential equation for the probability generating function $G(z, t)$, defined as

$$G(z, t) = \sum_n z^n \mathcal{P}_n(t).$$

Let us consider the birth-death process



subject to the initial condition of $n = n_0$ at $t = 0$. The aim of this mini-project is to derive the exact solution for the above process, using this method.

- (a) [5 marks] Show that the process has a mean population of $\bar{n}(t) = n_0 e^{(\lambda - \mu)t}$.
- (b) [10 marks] Derive the differential equation that governs the time evolution of the generating function of the above process, denoted as $G_{n_0}(z, t)$ henceforth.
- (c) [10 marks] State the boundary conditions and the initial conditions that are to be satisfied by $G_{n_0}(z, t)$.
- (d) [20 marks] Derive the exact solution for $G_{n_0}(z, t)$ subject to the above conditions, and express it in terms of $\bar{n}(t)$, μ , λ , and n_0 .
- (e) [15 marks] Derive the extinction probability $\mathcal{P}_0(t)$ as a function of time and discuss its limiting behaviour.
- (f) [15 marks] Calculate the first four cumulants of the distribution from the exact solution of $G_{n_0}(z, t)$, using a cumulant expansion, and express them in terms of t , μ , λ , and n_0 .
- (g) [10 marks] Find the long-time asymptotic limiting behaviour for the k -th cumulant, corresponding to the two regimes where $\lambda > \mu$ and $\lambda < \mu$.
- (h) [15 marks] Examine the special limit in which $\lambda = \mu$ in the exact solution $G_{n_0}(z, t)$, and derive the long-time limiting behaviour of the k -th cumulant in this case.