A15089W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

KINETIC THEORY

Hilary Term 2022

THURSDAY, 12th JANUARY 2023, 09:30 am to 12:30 pm

You should submit answers to all three questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. Consider a system of N indistinguishable particles, each of unit mass. Their positions $\mathbf{x}_i$ and velocities $\mathbf{v}_i$ for $i = 1, \ldots, N$ evolve according to

$$\frac{\mathrm{d}\mathbf{x}_i}{\mathrm{d}t} = \mathbf{v}_i, \qquad \frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t} = \sum_{j=1}^{N} K(\mathbf{x}_i, \mathbf{x}_j)(\mathbf{v}_j - \mathbf{v}_i).$$

The function K is symmetric and non-negative, $K(\mathbf{x}_i, \mathbf{x}_j) = K(\mathbf{x}_j, \mathbf{x}_i) \geqslant 0$ for all $\mathbf{x}_i$ and $\mathbf{x}_j$.

(a) [4 marks] Show that the total momentum of the system is conserved, and that the kinetic energy of the system decays according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\sum_{i=1}^{N}|\mathbf{v}_i|^2\right) = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} K(\mathbf{x}_i, \mathbf{x}_j)|\mathbf{v}_j - \mathbf{v}_i|^2.$$

(b) [3 marks] Now consider an ensemble of these N-particle systems. Write down the Liouville equation that expresses conservation of the probability density $\varrho(\mathbf{x}_1, \mathbf{v}_1, \ldots, \mathbf{x}_N, \mathbf{v}_N, t)$ of the ensemble in phase space.

(c) [7 marks] Use the Liouville equation and one further assumption to show that the one-particle distribution function $f(\mathbf{x}, \mathbf{v}, t)$ evolves according to

$$\partial_t f + \mathbf{v}\cdot\nabla f + \nabla_{\mathbf{v}}\cdot\left(f\mathsf{L}[f]\right) = 0, \qquad (\star)$$

where $\nabla_{\mathbf{v}}\cdot(\cdots)$ denotes a divergence with respect to $\mathbf{v}$, and the linear operator L is defined by

$$\mathsf{L}[f] = \int \mathrm{d}\mathbf{x}_\star \int \mathrm{d}\mathbf{v}_\star\, K(\mathbf{x}, \mathbf{x}_\star)(\mathbf{v}_\star - \mathbf{v}) f(\mathbf{x}_\star, \mathbf{v}_\star, t).$$

All integrals are taken over $\mathbb{R}^3$ and you may assume that f decays suitably at infinity. What additional assumption did you need to derive $(\star)$?

(d) [6 marks] Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int \mathrm{d}\mathbf{x}\int \mathrm{d}\mathbf{v}\,\frac{1}{2}|\mathbf{v}|^2 f\right) = -\int \mathrm{d}\mathbf{x}\int \mathrm{d}\mathbf{v}\int \mathrm{d}\mathbf{x}_\star\int \mathrm{d}\mathbf{v}_\star\, K(\mathbf{x}, \mathbf{x}_\star)|\mathbf{v}_\star - \mathbf{v}|^2 f(\mathbf{x}_\star, \mathbf{v}_\star, t) f(\mathbf{x}, \mathbf{v}, t),$$

and that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int \mathrm{d}\mathbf{x}\int \mathrm{d}\mathbf{v}\, f\log f\right) = \int \mathrm{d}\mathbf{x}\int \mathrm{d}\mathbf{x}_\star\, K(\mathbf{x}, \mathbf{x}_\star)\rho(\mathbf{x}_\star, t)\rho(\mathbf{x}, t).$$

Give an expression for the fluid density ρ, and also give a physical interpretation of the signs of the expressions on the right-hand sides.

(e) [5 marks] Show that the momentum density $\rho\mathbf{u}$ evolves according to

$$\partial_t(\rho\mathbf{u}) + \nabla\cdot\mathbf{\Pi} = \int \mathrm{d}\mathbf{x}_\star\, K(\mathbf{x}, \mathbf{x}_\star)\big(\mathbf{u}(\mathbf{x}_\star, t) - \mathbf{u}(\mathbf{x}, t)\big)\rho(\mathbf{x}_\star, t)\rho(\mathbf{x}, t).$$

Give expressions for the fluid velocity $\mathbf{u}$ and momentum flux $\mathbf{\Pi}$, and find an evolution equation for the fluid density ρ.

2. Consider a plasma consisting of spin-1/2 fermions deep in the range of temperatures where the gas is degenerate (examples of such situations are electron plasmas in metals or electron-hole plasmas in semiconductors; if you are unfamiliar with spin-1/2 fermions and/or Fermi-Dirac statistics, it does not matter, all the information about them that you will need to solve this question is provided below). The distributions of all species α ($\alpha = e$ or i denoting electrons or ions, respectively), in such a plasma can be approximated as completely degenerate, viz.,

$$f_{0\alpha}(\mathbf{v}) = \frac{1}{4}\left(\frac{m_\alpha}{\hbar\pi}\right)^3 H(v_{\mathrm{F}\alpha} - v), \tag{1}$$

where $\mathbf{v}$ is the velocity variable in three dimensions, m_α is the mass of the particles of species α, $m_\alpha v_{\mathrm{F}\alpha}^2/2$ is their Fermi energy, $v_{\mathrm{F}\alpha} = (3\pi^2 n_\alpha)^{1/3}\hbar/m_\alpha$, and $n_\alpha = \int d\mathbf{v} f_{0\alpha}(\mathbf{v})$ is their mean number density. $H(x)$ is the Heaviside function, equal to 1 if $x \geqslant 1$ and to 0 otherwise. Note that $H'(x) = \delta(x)$ (the Dirac delta).

(a) [7 marks] The dispersion relation that determines the complex increment p for the time evolution of linear perturbations in an electrostatic plasma is

$$\epsilon(p, \mathbf{k}) = 1 - \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{k^2}\frac{i}{n_\alpha}\int d\mathbf{v}\, \frac{1}{p + i\mathbf{k}\cdot\mathbf{v}}\mathbf{k}\cdot\frac{\partial f_{0\alpha}}{\partial \mathbf{v}} = 0, \tag{2}$$

where $\omega_{\mathrm{p}\alpha} = (4\pi q_\alpha^2 n_\alpha/m_\alpha)^{1/2}$ is the plasma frequency of species α, $\mathbf{k}$ is the wavenumber of the perturbation, and the integral with respect to the velocity component parallel to $\mathbf{k}$ is along the Landau contour. Let ω be the real frequency of the perturbation, and assume that $\omega > kv_{\mathrm{F}e}$ and $m_e \ll m_i$. Show that if the damping rate is either zero or small compared to the frequency ω, then the latter satisfies the following equation:

$$1 + \frac{3\omega_{\mathrm{pe}}^2}{k^2 v_{\mathrm{Fe}}^2}\left[1 + \frac{\omega}{2kv_{\mathrm{Fe}}}\ln\left(\frac{\omega - kv_{\mathrm{Fe}}}{\omega + kv_{\mathrm{Fe}}}\right)\right] = 0. \tag{3}$$

(b) [3 marks] Do you expect these waves to be Landau-damped? If yes, explain how the rate of this damping should be determined. If not, explain why not.

(c) [5 marks] In the long-wavelength limit $k\lambda_{\mathrm{De}} \ll 1$, where $\lambda_{\mathrm{De}} = \sqrt{3}v_{\mathrm{F}e}/\omega_{\mathrm{pe}}$ is the Debye length, derive the expression for the frequencies of Langmuir waves in a degenerate plasma, including the lowest-order dispersive (i.e., k-dependent) term.

(d) [5 marks] In the short-wavelength limit $k\lambda_{\mathrm{De}} \gg 1$, show that the dispersion relation (3) has a solution of the form

$$\omega = \pm kv_{\mathrm{Fe}}\left(1 + \delta\right), \tag{4}$$

where δ is a small correction. Find δ. This solution is called "zero-point sound".

(e) [5 marks] Sketch ω vs. k in a degenerate plasma. What is the main difference between this and the situation in a non-degenerate plasma? Without derivations, but explaining your reasoning, make an educated guess about what other propagating waves one might hope to find in a degenerate electron-ion plasma and what their damping properties might be.

3. (a) [2 marks] Consider two arbitrary smooth functions of angle-action coordinates, $g(\boldsymbol{\theta}, \mathbf{J})$ and $h(\boldsymbol{\theta}, \mathbf{J})$. Using these coordinates write down the expression for the Poisson bracket $[g, h]$.

(b) [6 marks] Let f be the distribution function of an ensemble of particles whose motion is governed by the 'mean field + perturbation' Hamiltonian

$$H(\boldsymbol{\theta}, \mathbf{J}, t) = H_0(\mathbf{J}) + \delta\Phi(\boldsymbol{\theta}, \mathbf{J}, t). \tag{5}$$

The equation governing the evolution of f is

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + [f, H] = 0. \tag{6}$$

Let $f(\boldsymbol{\theta}, \mathbf{J}, t) = f_0(\mathbf{J}) + \delta f(\boldsymbol{\theta}, \mathbf{J}, t)$, where $f_0(\mathbf{J})$ is the unperturbed DF. Fourier expanding the potential as $\delta\Phi = \sum_{\mathbf{k}} \delta\Phi_{\mathbf{k}}(\mathbf{J}, t) \exp(i\mathbf{k}\cdot\boldsymbol{\theta})$ and similarly for δf, where $\mathbf{k} \in \mathbb{Z}^3$, assuming all perturbations are small, and assuming that $\delta\Phi$ is switched on at $t = 0$ and that $\delta f_{\mathbf{k}}(\mathbf{J}, 0) = 0$, show that the linear response of the DF satisfies

$$\delta f_{\mathbf{k}}(\mathbf{J}, t) = i\mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{J}} \int_0^t \mathrm{d}t' \delta\Phi_{\mathbf{k}}(\mathbf{J}, t') \mathrm{e}^{-i\mathbf{k}\cdot\boldsymbol{\Omega}(\mathbf{J})(t-t')}, \tag{7}$$

where you should define the frequency vector $\boldsymbol{\Omega}(\mathbf{J})$.

(c) [6 marks] Consider a galaxy consisting of an initially spherically symmetric dark matter halo, plus a rigidly rotating 'bar' of stars which is centred at the origin and which rotates anticlockwise about the z axis. In the absence of the bar, the dynamics of a dark matter particle is governed by the Hamiltonian $H_0(\mathbf{J})$. Let the bar perturbation have potential $\delta\Phi$. Each dark matter particle then moves in the time-dependent Hamiltonian $H = H_0 + \delta\Phi$. Let the dark matter distribution function be f, normalized such that $\int \mathrm{d}\boldsymbol{\theta}\mathrm{d}\mathbf{J} f = 1$. Suitable angle-action coordinates for describing this system are

$$\boldsymbol{\theta} = (\theta_r, \theta_\psi, \theta_\varphi), \qquad \mathbf{J} = (J_r, L, L_z), \tag{8}$$

where in particular, J_r is the radial action, L is the specific angular momentum, and L_z is the z-component of specific angular momentum. Write down Hamilton's equation for the evolution of L_z. Hence show that the total torque induced by the bar upon the halo, per unit halo mass, is equal to

$$\mathcal{T}(t) = -\int \mathrm{d}\boldsymbol{\theta}\mathrm{d}\mathbf{J}\, f(\boldsymbol{\theta}, \mathbf{J}, t) \frac{\partial \delta\Phi(\boldsymbol{\theta}, \mathbf{J}, t)}{\partial \theta_\varphi}. \tag{9}$$

Making the same expansions as in part (a), show that

$$\mathcal{T}(t) = \sum_{\mathbf{k}} i(2\pi)^3 k_\varphi \int \mathrm{d}\mathbf{J}\, \delta f_{\mathbf{k}}(\mathbf{J}, t) \delta\Phi^*_{\mathbf{k}}(\mathbf{J}, t), \tag{10}$$

where $\mathbf{k} = (k_r, k_\psi, k_\varphi)$ are integer vectors. You may quote the identity $\int \mathrm{d}\boldsymbol{\theta} \exp(i\mathbf{k}\cdot\boldsymbol{\theta}) = (2\pi)^3 \delta^{\mathbf{0}}_{\mathbf{k}}$.

(d) [8 marks] Let the bar rotate anticlockwise with fixed angular speed $\Omega_{\mathrm{p}} > 0$; then we can write $\delta\Phi_{\mathbf{k}}(\mathbf{J}, t) = \Psi_{\mathbf{k}}(\mathbf{J}) \exp(-ik_\varphi \Omega_{\mathrm{p}} t)$. By combining this with equations (7) and (10), and stating any further assumptions you make, show that in the limit $t \to \infty$, the torque on the dark matter halo reaches a steady-state value:

$$\mathcal{T} = -\sum_{\mathbf{k}} \pi(2\pi)^3 k_\varphi \int \mathrm{d}\mathbf{J} |\Psi_{\mathbf{k}}|^2 \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{J}} \delta\left(\mathbf{k}\cdot\boldsymbol{\Omega} - k_\varphi \Omega_{\mathrm{p}}\right). \tag{11}$$

(e) [3 marks] Suppose $f_0(\mathbf{J})$ depends on $\mathbf{J}$ only through the mean field energy, i.e. $f_0 = f_0(E)$ where $E = H_0(\mathbf{J})$, and that $\mathrm{d}f_0/\mathrm{d}E < 0$. Show that in this case $\mathcal{T}$ is positive definite. What might this result imply about the long-term evolution of stellar bars? (You may assume the bar has a positive moment of inertia.)