A15274W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

GEOPHYSICAL FLUID DYNAMICS

Trinity Term 2024

Wednesday 5 June 2024, 14:30 - 16:30

You should submit answers to both questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. (a) [6 marks] Briefly explain the origin of geostrophic balance, as given by

$$fv = \frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad fu = -\frac{1}{\rho}\frac{\partial p}{\partial y},$$

where the symbols have their usual meanings. Suppose that rapid atmospheric convection leads to a localised surface pressure anomaly which is zero apart from $p' = -20\,\text{hPa}$ within a radius of 500 km of a central point. By referring to geostrophic balance, how can you tell that this anomaly will be unstable? The resulting flow changes are termed *geostrophic adjustment*. After this adjustment, how should the pressure anomaly be reduced in order to keep the wind field of order $10\,\text{ms}^{-1}$ over the same length scale? Based on this, why are tropical temperature gradients weaker than those in mid-latitudes?

(b) [6 marks] Geostrophic adjustment involves the emission of internal gravity waves to remove energy from the perturbation. Given the hydrostatic, non-rotating form of the dispersion relation

$$\omega^2 = \frac{N^2 k^2}{m^2},$$

determine and sketch the direction and speed of the resulting waves. Here N is the buoyancy frequency and k and m are the horizontal and vertical wavenumbers respectively.

(c) [6 marks] Equatorial Kelvin waves satisfy the following equations:

$$\frac{\partial u}{\partial t} + g'\frac{\partial h}{\partial x} = 0, \qquad \beta y u + g'\frac{\partial h}{\partial y} = 0, \qquad \frac{\partial h}{\partial t} + h_0\frac{\partial u}{\partial x} = 0,$$

where h is the thickness of the surface ocean layer. List the assumptions that have been made in deriving these. Given that a wavelike solution exists with

$$h = A_0\, e^{-y^2/L_{eq}^2}\, e^{i(kx-\omega t)},$$

where L_{eq} is the equatorial deformation radius, sketch the height and velocity fields at an instant in time in the (x, y) plane. Describe how the flow relates to geostrophic balance in both zonal and meridional directions.

(d) [4 marks] Quasi-geostrophic Rossby waves in a continuously stratified flow satisfy the dispersion relation

$$\omega = Uk - \frac{\beta k}{k^2 + \alpha^2},$$

where $\alpha^2 = l^2 + f_0^2 m^2/N^2$ and m is the vertical wavenumber. Show that stationary waves (i.e. with zero phase velocity) must have an eastward zonal group velocity.

(e) [3 marks] State one example each of how Kelvin, gravity and Rossby waves contribute to El Niño events such as that in the 2023/24 winter.

2. (a) [5 marks] The equation of motion for a two-dimensional ocean, forced by a zonal wind stress τ_s and dissipated by linear friction, can be written as:

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + 2\boldsymbol{\Omega} \times \mathbf{u} + \frac{1}{\rho_0}\nabla p = \left(\frac{\tau_s}{\rho_0 H}, 0\right) - r\mathbf{u},$$

where the symbols have their usual meanings.
Derive the vorticity equation

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\zeta = -\frac{1}{\rho_0 H}\frac{\partial \tau_s}{\partial y} - r\xi,$$

where ζ is the absolute vorticity and ξ is the relative vorticity, stating any assumptions you have made.

(b) [10 marks] Write down the form of the vorticity equation known as "Sverdrup balance", which holds approximately in the interior of an ocean gyre. List the conditions required for its validity. Assuming that the wind stress varies only with latitude, show that the northward volume transport predicted by Sverdrup balance is

$$T = -\frac{L}{\beta\rho_0}\frac{\partial \tau_s}{\partial y},$$

where L is the basin width and β has its usual meaning. Sketch the circulation predicted by Sverdrup balance in a typical hemispheric ocean basin, stating clearly any boundary conditions that you assume. Estimate the volume transport of a typical subtropical gyre, assuming a maximum wind stress of magnitude $0.1\,\mathrm{N\,m^{-2}}$ and typical values for the remaining parameters.

(c) [10 marks] Suppose that the gyre is closed by a frictional boundary current of width δ. By scaling the appropriate terms in the vorticity equation, or otherwise, deduce that

$$\delta \sim \frac{r}{\beta}.$$

Assuming a frictional time scale of 1 year and $H = 1\,\mathrm{km}$, estimate δ and a typical boundary current velocity. Comment on the extent to which these values are realistic. Using these values, estimate the magnitude of the relative vorticity in the boundary current, and thus deduce that it cannot be neglected in reality.