A15089W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

COLLISIONLESS PLASMA PHYSICS

Trinity Term 2023

You should submit answers to all three questions.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

In the following two questions, we consider a plasma consisting of electrons with mass m_e, charge $-e$, and number density n_e and ions with mass m_i, charge Ze, and number density n_i. Assume that $Z \sim 1$ and $m_e \ll m_i$.

1. Consider a homogeneous two-species plasma (as outlined above) with a uniform equilibrium magnetic field $\mathbf{B}_0 = B\hat{\mathbf{z}}$.

   (a) [5 marks] Starting from Maxwell's equations, show that a plane-wave perturbation of the form

   $$\delta\mathbf{E} = \tilde{\mathbf{E}}e^{-i\omega t+i\mathbf{k}\cdot\mathbf{r}}, \quad \delta\mathbf{B} = \tilde{\mathbf{B}}e^{-i\omega t+i\mathbf{k}\cdot\mathbf{r}}, \quad \delta\mathbf{J} = \tilde{\mathbf{J}}e^{-i\omega t+i\mathbf{k}\cdot\mathbf{r}}$$

   satisfies

   $$\left[n^2(\hat{\mathbf{k}}\hat{\mathbf{k}} - \mathbf{I}) + \boldsymbol{\epsilon}\right]\cdot\tilde{\mathbf{E}} = 0,$$

   where the dielectric tensor can be written

   $$\boldsymbol{\epsilon} = \mathbf{I} + \frac{i\boldsymbol{\sigma}}{\epsilon_0\omega}.$$

   Define the index of refraction n, the conductivity tensor $\boldsymbol{\sigma}$, and the dispersion relation. Write down the components of the cold-plasma dielectric tensor $\boldsymbol{\epsilon}$ in the right-handed orthonormal basis $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ where $\hat{\mathbf{x}} = \mathbf{k}/k$, making sure to define any symbols you use.

   (b) [5 marks] Assuming $\Omega_e \sim \omega_{\rm pe}$ and $\sqrt{\Omega_i\Omega_e} \ll \omega \ll \Omega_e$, where the symbols $\omega_{\rm pe}, \omega_{\rm pi}, \Omega_e, \Omega_i$ have their usual meaning, show that

   $$\Omega_i \ll \omega_{\rm pi} \ll \omega \ll \Omega_e$$

   and hence that, to lowest order, the cold-plasma dielectric tensor is given by

   $$\boldsymbol{\epsilon} = \begin{pmatrix} 1+\omega_{\rm pe}^2/\Omega_e^2 & -i\omega_{\rm pe}^2/\Omega_e\omega & 0 \\ i\omega_{\rm pe}^2/\Omega_e\omega & 1+\omega_{\rm pe}^2/\Omega_e^2 & 0 \\ 0 & 0 & -\omega_{\rm pe}^2/\omega^2 \end{pmatrix}. \tag{1}$$

   Carefully justify any terms that you neglect.

   (c) [10 marks] If $n^2 \sim \omega_{\rm pe}/\omega$, show that $kd_e \ll 1$, where $d_e = c/\omega_{\rm pe}$. Show that cold-plasma dispersion relation for the dielectric tensor (1) gives

   $$\omega \approx k_\parallel k d_e^2 \Omega_e. \tag{2}$$

   (d) [5 marks] A radio hobbyist tunes their radio to 1–5 kHz and observes multiple signals whose frequency decays monotonically from 5 kHz down to 1 kHz over a duration of 1 s. They conjecture that these signals are generated as broadband noise on the other side of the planet and propagate as waves along Earth's magnetic field according to (2) and at an altitude comparable to Earth's radius. Approximating Earth's magnetic field as $B \sim 10^{-5}$ T and Earth's radius as 6400 km, estimate the density n_e of the plasma trapped in Earth's magnetic field.

2. Consider an inhomogeneous two-species plasma (as outlined above), with electron density $n_e(r)$ confined in a θ-pinch with magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$, where $\{r, \varphi, z\}$ are the usual cylindrical polar coordinates.

   (a) [5 marks] What is the WKB approximation for a wave of characteristic wavelength λ propagating through this plasma? Define the eikonal function S, the spatially varying wavenumber $\mathbf{k}$, and carefully state all ordering assumptions. Show that, to lowest order in the appropriate expansion parameter, the frequency ω and wavenumber $\mathbf{k}$ satisfy the cold-plasma dispersion relation at every point in space.

   (b) [10 marks] Define the group velocity $\boldsymbol{v}_\mathrm{g}(\mathbf{r}, \mathbf{k})$ and derive the ray-tracing equation

$$\boldsymbol{v}_\mathrm{g}\cdot\nabla\mathbf{k} = -\left.\frac{\partial\omega}{\partial\mathbf{r}}\right|_{\mathbf{k}}.$$

   Let us parametrise the position $\mathbf{r}$ along a ray as $\mathbf{r}(\tau)$ where

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\tau} = \boldsymbol{v}_\mathrm{g}.$$

   Show that $\mathbf{r}(\tau)$ and $\mathbf{k}(\tau)$ along the ray satisfy the equations

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\tau} = \left.\frac{\partial H}{\partial\mathbf{k}}\right|_{\mathbf{r}},$$
$$\frac{\mathrm{d}\mathbf{k}}{\mathrm{d}\tau} = -\left.\frac{\partial H}{\partial\mathbf{r}}\right|_{\mathbf{k}},$$

   for an appropriate $H(\mathbf{r}, \mathbf{k})$ that you must determine.

   (c) [5 marks] The O-mode satisfies $\omega^2 = \omega_\mathrm{pe}^2 + k^2c^2$, where $\omega_\mathrm{pe}^2 = e^2 n_e/\epsilon_0 m_e$. Show that the position $\mathbf{r}(\tau)$ along an O-mode ray satisfies

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}\tau^2} = -\frac{\mathrm{d}V}{\mathrm{d}\mathbf{r}},$$

   where $V(r) = c^2\omega_\mathrm{pe}(r)^2/2\omega^2$.

   (d) [15 marks] Consider a ray launched towards the plasma from the outside at $r = d$ and at an angle α from the centre of the θ-pinch. Working in cylindrical polar coordinates, find an equation for $\mathrm{d}^2 r/\mathrm{d}\tau^2$ and thus deduce that the radial distance r_0 at which the ray reflects is determined by

$$\frac{\omega_\mathrm{pe}^2(r_0)}{\omega^2} = 1 - \left(\frac{d^2}{r_0^2} - 1\right)\sin^2\alpha.$$

3. Consider a plasma that obeys the Kinetic Magnetohydrodynamics (KMHD) approximation. In what follows, all symbols have their usual meaning. Thus, α is the species index, the subscripts $\perp$ and $\|$ refer to the directions perpendicular and parallel, respectively, to the local direction $\mathbf{b}$ of the magnetic field $\mathbf{B} = B\mathbf{b}$, $\mathbf{w} = \mathbf{v} - \mathbf{u}_\alpha$ is the peculiar velocity of the particles, and $\mathbf{u}_\alpha$ their mean velocity (in general, different for different species). You may quote and use any result from the Lecture Notes without derivation. Some formulae that may prove useful are listed on the next page.

(a) [7 marks] By any method you prefer, show that the gyroaveraged distribution function $f_\alpha(t, \mathbf{r}, \mu, \varepsilon)$, expressed in the variables $\mu = w_\perp^2/2B$ (magnetic moment) and $\varepsilon = m_\alpha w^2/2$ (energy), satisfies (to lowest order in the high-flow regime),

$$\frac{Df_\alpha}{Dt_\alpha} + m_\alpha \left[w_\| \left(\frac{q_\alpha}{m_\alpha} E_\| - \frac{D\mathbf{u}_\alpha}{Dt_\alpha} \cdot \mathbf{b} \right) + \mu \frac{\mathrm{d}B}{\mathrm{d}t_\alpha} \right] \frac{\partial f_{0\alpha}}{\partial \varepsilon} = 0, \tag{3}$$

$$\text{where} \quad \frac{D}{Dt_\alpha} = \frac{\mathrm{d}}{\mathrm{d}t_\alpha} + w_\| \mathbf{b} \cdot \boldsymbol{\nabla}, \quad \frac{\mathrm{d}}{\mathrm{d}t_\alpha} = \frac{\partial}{\partial t} + \mathbf{u}_\alpha \cdot \boldsymbol{\nabla}. \tag{4}$$

(b) [8 marks] Hence, or otherwise, show that the total kinetic energy of the particle's peculiar motion, $\mathscr{K} = \sum_\alpha \iint \mathrm{d}\mathbf{r}\mathrm{d}\mathbf{w}\, (m_\alpha w^2/2) f_\alpha$, satisfies

$$\frac{\mathrm{d}\mathscr{K}}{\mathrm{d}t} = -\int \mathrm{d}\mathbf{r} \sum_\alpha \mathsf{P}[f_\alpha] : \boldsymbol{\nabla}\mathbf{u}_\alpha, \quad \mathsf{P}[f_\alpha] = p_\perp[f_\alpha]\,(\mathsf{I} - \mathbf{b}\mathbf{b}) + p_\|[f_\alpha]\mathbf{b}\mathbf{b}, \tag{5}$$

where $p_\perp[f_\alpha] = \int \mathrm{d}\mathbf{w}\, (m_\alpha w_\perp^2/2) f_\alpha$ and $p_\|[f_\alpha] = \int \mathrm{d}\mathbf{w}\, m_\alpha w_\|^2 f_\alpha$ are the perpendicular and parallel pressures, respectively. Interpret this result physically.

(c) [8 marks] Assume that the distribution consists of a spatially homogeneous mean and a small perturbation: $f_\alpha = f_{0\alpha}(\mu, \varepsilon) + \delta f_\alpha(t, \mathbf{r}, \mu, \varepsilon)$ and $\mathbf{B} = B_0 \hat{\mathbf{z}} + \delta\mathbf{B}(t, \mathbf{r})$. Use the kinetic equation (3), the exact result (5), and the conservation of the total energy of the plasma (which you need not prove), to show that, to lowest order in perturbations,

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[\mathscr{A} + \int \mathrm{d}\mathbf{r} \left(\sum_\alpha \frac{m_\alpha n_\alpha u_\alpha^2}{2} + \frac{B^2}{8\pi} \right) \right] = \int \mathrm{d}\mathbf{r} \sum_\alpha \mathsf{P}[f_{0\alpha}] : \boldsymbol{\nabla}\mathbf{u}_\alpha, \tag{6}$$

$$\text{where} \quad \mathscr{A} = \sum_\alpha \iint \mathrm{d}\mathbf{r}\mathrm{d}\mathbf{w}\, \frac{\delta f_\alpha^2}{2(-\partial f_{0\alpha}/\partial \varepsilon)}. \tag{7}$$

Note that, since $f_{0\alpha}$ is a function of μ, and the exact B is involved in the definition of μ, the pressure tensor $\mathsf{P}[f_{0\alpha}] = \mathsf{P}[f_\alpha] - \mathsf{P}[\delta f_\alpha]$ contains both the mean pressure and a part of the perturbed pressure (the "non-resonant" part). *Hint:* To work out this perturbed pressure, one expands μ inside $f_{0\alpha}(\mu, \varepsilon)$ in small $\delta B/B_0$.

(d) [10 marks] Work out $p_\perp[f_{0\alpha}]$ and $p_\|[f_{0\alpha}]$, and hence show that the "generalised free energy" is conserved to lowest non-trivial order in the perturbations:

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} = 0, \quad \text{where} \quad \mathscr{F} = \mathscr{A} + \int \mathrm{d}\mathbf{r} \left\{ \sum_\alpha \frac{m_\alpha n_\alpha u_\alpha^2}{2} + \left[1 - \sum_\alpha \frac{\beta_{\|\alpha}}{2} \left(1 - \frac{p_{\perp\alpha}}{p_{\|\alpha}} \right) \right] \frac{\delta B_\perp^2}{8\pi} \right.$$
$$\left. + \left[1 - \sum_\alpha \beta_{\perp\alpha} \left(\frac{p_{\perp\alpha}}{p_{\|\alpha}} C_\alpha - 1 \right) \right] \frac{\delta B_\|^2}{8\pi} \right\}, \tag{8}$$

where $p_{\perp\alpha}$, $p_{\|\alpha}$, $\beta_{\perp\alpha} = 8\pi p_{\perp\alpha}/B_0^2$, $\beta_{\|\alpha} = 8\pi p_{\|\alpha}/B_0^2$ are all mean quantities, which can be taken to be constant in time and space, and the constant C_α is defined by

$$\frac{2p_{\perp\alpha}^2}{p_{\|\alpha}} C_\alpha = -2\pi m_\alpha^2 B_0^2 \iint \mathrm{d}\varepsilon \mathrm{d}\mu\, J \mu^2 \frac{\partial f_{0\alpha}}{\partial \varepsilon} \tag{9}$$

($C_\alpha = 1$ if $f_{0\alpha}$ is a bi-Maxwellian distribution). **See next page for part (e).**

(e) [7 marks] What does the conservation of $\mathscr{F}$ imply for the stability of small perturbations in KMHD? Use (8) to formulate a sufficient condition of stability. How is this result compatible with the existence (or otherwise) of firehose and mirror instabilities?

Useful formulae:

$$\int \mathrm{d}\mathbf{w} = 2\pi \iint \mathrm{d}w_\parallel \mathrm{d}w_\perp w_\perp = 2\pi \iint \mathrm{d}\varepsilon \mathrm{d}\mu\, J, \quad J = \frac{B}{m_\alpha |w_\parallel|}, \quad |w_\parallel| = \sqrt{\frac{2\varepsilon}{m_\alpha} - 2\mu B}, \tag{10}$$

$$\frac{\partial J}{\partial t} + \nabla \cdot \langle \dot{\mathbf{r}} \rangle J + \frac{\partial}{\partial \varepsilon} \langle \dot{\varepsilon} \rangle J = 0 \quad \text{(conservation of phase-space volume)}, \tag{11}$$

$$\int \mathrm{d}\mathbf{w}\, w_\parallel f_\alpha = 0 \quad \text{(defnition of peculiar velocity)}, \tag{12}$$

$$\left(\frac{\partial f_{0\alpha}}{\partial \mu} \right)_\varepsilon = B \left[\frac{1}{w_\perp} \left(\frac{\partial f_{0\alpha}}{\partial w_\perp} \right)_{w_\parallel} - \frac{1}{w_\parallel} \left(\frac{\partial f_{0\alpha}}{\partial w_\parallel} \right)_{w_\perp} \right], \tag{13}$$

$$\frac{\mathrm{d} \ln B}{\mathrm{d}t_\alpha} = \mathbf{bb} : \nabla \mathbf{u}_\alpha - \nabla \cdot \mathbf{u}_\alpha = -(\mathsf{I} - \mathbf{bb}) : \nabla \mathbf{u}_\alpha \quad \text{(the induction equation)}, \tag{14}$$

$$\ln B = \ln \sqrt{B_0^2 + 2\mathbf{B}_0 \cdot \delta\mathbf{B} + |\delta\mathbf{B}|^2} \approx \ln B_0 + \frac{\delta B_\parallel}{B_0} + \frac{\delta B_\perp^2 - \delta B_\parallel^2}{2B_0^2} \quad \text{(Taylor expansion)}. \tag{15}$$