A15758W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

ADVANCED QUANTUM THEORY: PATH INTEGRALS AND MANY-PARTICLE PHYSICS

Trinity Term 2024

Tuesday 4th June 14:30 pm - 16:30 pm

You should submit answers to both questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. [25 marks] Consider the quantum harmonic oscillator with mass m, frequency ω and Hamiltonian H at inverse temperature β, and let $|x\rangle$ denote an eigenstate of the position operator with position coordinate x. A diagonal matrix element of the Boltzmann factor has the path-integral representation

$$\langle x|e^{-\beta H}|x\rangle = \int \mathcal{D}x(\tau)\, e^{-S[x(\tau)]/\hbar} \tag{1}$$

where the Euclidean or imaginary-time action is

$$S[x(\tau)] = \int_0^{\beta\hbar} \mathrm{d}\tau \left[\frac{m}{2}\left(\frac{\mathrm{d}x(\tau)}{\mathrm{d}\tau}\right)^2 + \frac{m\omega^2}{2}(x(\tau))^2\right] \tag{2}$$

and the functional integral is over paths $x(\tau)$ with end-points $x(0) = x(\beta\hbar) = x$.

(a) [8 marks] This action is stationary on the path $x_0(\tau)$. Use calculus of variations to find a differential equation satisfied by $x_0(\tau)$, solve this equation and hence show that the action for this path is

$$S[x_0(\tau)] = m\omega x^2 \tanh(\beta\hbar\omega/2)\,. \tag{3}$$

(b) [7 marks] Use the change of variables $x(\tau) = x_0(\tau) + y(\tau)$ in Eqns. 1 and 2 to show that

$$\langle x|e^{-\beta H}|x\rangle = e^{-S[x_0(\tau)]/\hbar} Z\,, \tag{4}$$

giving an expression for Z as a path integral. What is the dependence of the factors $e^{-S[x_0(\tau)]}$ and Z on the value of x?

(c) [5 marks] Find expressions for the ratio

$$R = \frac{\langle x|e^{-\beta H}|x\rangle}{\langle y|e^{-\beta H}|y\rangle|_{y=0}}$$

in the limits $\beta\hbar\omega \ll 1$ and $\beta\hbar\omega \gg 1$. Suggest a way of understanding your result for R at high temperature.

(d) [5 marks] Let $\varphi_n(x)$ denote the normalised eigenfunctions of H with eigenvalues $(n+\frac{1}{2})\hbar\omega$ for $n = 0, 1, 2\ldots$. Give an expression for $\langle x|e^{-\beta H}|x\rangle$ in terms of $\varphi_n(x)$. In the light of this expression, discuss your result for R at low temperature.

2. [25 marks] N identical bosons, each of mass m, move in a cubic box of side L with periodic boundary conditions. Single-particle momentum eigenstates with wavevector $\mathbf{k}$ have position-space wavefunctions $\langle \mathbf{r}|\mathbf{k}\rangle = L^{-3/2}e^{i\mathbf{k}\cdot\mathbf{r}}$ with $\mathbf{k} = 2\pi(\ell_x, \ell_y, \ell_z)/L$ where ℓ_x, ℓ_y and ℓ_z are integer. The creation operator for a boson in a momentum eigenstate is $\psi^\dagger_{\mathbf{k}}$ and the annihilation operator is $\psi_{\mathbf{k}}$. They obey the commutation relations $[\psi_{\mathbf{k}}, \psi^\dagger_{\mathbf{q}}] = \delta_{\mathbf{k},\mathbf{q}}$ and $[\psi_{\mathbf{k}}, \psi_{\mathbf{q}}] = 0$. The position-space creation operator is $\psi^\dagger(\mathbf{r}) = L^{-3/2}\sum_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{r}}\psi^\dagger_{\mathbf{k}}$. The Hamiltonian for the system is $H = H_{\rm KE} + H_{\rm int}$ with

$$H_{\rm KE} = \frac{\hbar^2}{2m}\sum_{\mathbf{k}} |\mathbf{k}|^2 \psi^\dagger_{\mathbf{k}}\psi_{\mathbf{k}} \quad \text{and} \quad H_{\rm int} = \frac{u}{2}\int \mathrm{d}^3\mathbf{r}\, \psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r})\psi(\mathbf{r})\psi(\mathbf{r})\,. \tag{1}$$

New operators $\varphi^\dagger_{\mathbf{k}}$ and $\varphi_{\mathbf{k}}$ are defined for $|\mathbf{k}| \neq 0$ by the transformation

$$\varphi^\dagger_{\mathbf{k}} = c_k\, \psi^\dagger_{\mathbf{k}} + s_k\, \psi_{-\mathbf{k}} \quad \text{with inverse} \quad \psi^\dagger_{\mathbf{k}} = c_k\, \varphi^\dagger_{\mathbf{k}} - s_k\, \varphi_{-\mathbf{k}}\,, \tag{2}$$

where $c_k = \cosh\theta_k$, $s_k = \sinh\theta_k$ and θ_k is a real function of $k \equiv |\mathbf{k}|$. The state $|0\rangle$ has the following properties: it is annihilated by $\varphi_{\mathbf{k}}$ for all $|\mathbf{k}| \neq 0$ (so that $\varphi_{\mathbf{k}}|0\rangle = 0$) and it is an eigenstate of ψ_0 with real eigenvalue $\sqrt{N_0}$ (so that $\psi_0|0\rangle = \sqrt{N_0}|0\rangle$).

(a) [5 marks] Write the total boson number operator $\hat{N}$ and the Hamiltonian contribution $H_{\rm int}$ in terms of $\psi_{\mathbf{k}}$ and $\psi^\dagger_{\mathbf{k}}$.

(b) [4 marks] Show that $[\varphi_{\mathbf{k}}, \varphi^\dagger_{\mathbf{q}}] = \delta_{\mathbf{k},\mathbf{q}}$ and $[\varphi_{\mathbf{k}}, \varphi_{\mathbf{q}}] = 0$.

(c) [6 marks] Suppose that N_0 is large and consider an expansion of $\langle 0|H_{\rm int}|0\rangle$ in decreasing powers of N_0. Show that

$$\langle 0|H_{\rm int}|0\rangle = \frac{uN_0^2}{2L^3} + \frac{uN_0}{2L^3}\sum_{\mathbf{k}\neq 0}\Big[\langle 0|\psi_{\mathbf{k}}\psi_{-\mathbf{k}}|0\rangle + \langle 0|\psi^\dagger_{\mathbf{k}}\psi^\dagger_{-\mathbf{k}}|0\rangle + 4\langle 0|\psi^\dagger_{\mathbf{k}}\psi_{\mathbf{k}}|0\rangle\Big] + \mathcal{O}([N_0]^0)$$

where $\mathcal{O}([N_0]^0)$ denotes terms independent of N_0 which you may omit. Evaluate the expectation values appearing in this expression in terms of θ_k.

(d) [5 marks] Evaluate $\langle 0|\hat{N}|0\rangle$ in terms of N_0 and θ_k. Hence suggest an expression for N_0 in terms of N and θ_k and show (again omitting terms independent of N_0) that

$$\langle 0|H_{\rm int}|0\rangle \approx \frac{u\rho}{2}N + u\rho\sum_{\mathbf{k}\neq 0}(\sinh^2\theta_k - \sinh\theta_k\cosh\theta_k)\,.$$

where $\rho \equiv N/L$ is the boson number density.

(e) [5 marks] Evaluate $\langle 0|H_{\rm KE}|0\rangle$ and show that $\langle 0|H|0\rangle$ is minimised by the choice

$$\tanh 2\theta_k = \frac{u\rho}{u\rho + \hbar^2k^2/2m}\,.$$

Discuss physically why $\langle 0|H|0\rangle$ is lower for suitable non-zero $\theta_{\mathbf{k}}$ than with $\theta_{\mathbf{k}} = 0$ for all $\mathbf{k}$.