Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

ADVANCED QUANTUM FIELD THEORY FOR PARTICLE PHYSICS

Trinity Term 2024

Tuesday, 16th April 2024, 9:30 am to 12:30 pm

You should submit answers to all three questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. The Dirac Lagrangian is

$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi \ ,$$

where ψ is a 4-component Dirac field.

(a) [2 marks] Write down the path integral representation of the partition function for the Dirac theory. Show that the Dirac Lagrangian is invariant under a $U(1)$ global symmetry, $\psi \to e^{i\alpha}\psi$, where α is a constant.

(b) [4 marks] Calculate the change in the Dirac action under a local transformation, $\psi \to e^{i\alpha(x)}\psi$ for an infinitesimal transformation. Identify the Noether currents.

(c) [5 marks] Define what an operator equation is in the path integral language. Show that the Noether current is conserved as an operator equation.

(d) [5 marks] Write down the partition function for the Dirac theory with a source (also called a background gauge field) included for the current operator. Show that the partition function is invariant under a gauge transformation of the background gauge field.

(e) [5 marks] Consider a theory of two Dirac fermions with equal masses. Write down the Lagrangian for the theory. What is the global internal symmetry of this Lagrangian? Write down the corresponding Noether current.

(f) [4 marks] Write the Lagrangian for the two massive Dirac fermions in terms of Weyl fermions. What is the symmetry of the Lagrangian when all the masses are equal to zero?

2. The Lagrangian for QED is

$$\mathcal{L} = i\bar{\psi}\not{D}\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \ ,$$

where ψ is a 4-component Dirac field, A_μ is the photon, e is the electromagnetic coupling and $D_\mu = \partial_\mu + ieA_\mu$.

(a) [4 marks] Draw the one-loop Feynman diagrams for the amputated 3-point photon amplitude, $\mathcal{M}^{\mu\nu\rho}$. Show that the diagrams can be made identical with only the arrow on the fermion line flipped.

(b) [4 marks] Write down the amplitude for each of the diagrams. You do not have to simplify the expression yet.

(c) [6 marks] Show that the one-loop contribution to the photon 3-point function vanishes. You may find the identity $C\gamma^\mu C^{-1} = (-\gamma^\mu)^\intercal$ to be useful, where C is associated with charge conjugation, but its explicit form will not be required.

(d) [4 marks] Argue that the sum of one-loop diagrams contributing to any n-point photon amplitude vanishes pairwise when n is odd.

(e) [4 marks] Draw the one-loop Feynman diagrams for the 4-point photon amplitude, $\mathcal{M}^{\mu\nu\rho\sigma}$. For diagrams related by permutation of external legs it will be sufficient to specify the permutations. You do not have to compute the amplitude.

(f) [3 marks] Argue that the 4-point amplitude is finite using gauge invariance and power counting.

Some useful information

- Photon-fermion vertex $= -ie\gamma^\mu$.
- Fermion propagator $= i(\not{p} + m)/(p^2 - m^2)$.

3. Consider a non-Abelian gauge theory with a gauge group $SU(2)$, with a single Dirac fermion ψ transforming under the fundamental representation of $SU(2)$.

   (a) [5 marks] Write down the Lagrangian for the theory, including the gauge fixing and ghost terms in the R_ξ gauge.

   (b) [3 marks] Consider the scattering process of fermion-antifermion annihilating to two gauge bosons,

   $$\psi(p_1)\bar{\psi}(p_2) \to g(k_1)g(k_2) \ .$$

   Draw the tree-level Feynman diagrams contributing to the process.

   (c) [6 marks] Write down the amputated amplitude $\mathcal{M}^{\mu\nu}$ (more precisely, including external leg factors for the fermions but not for the gauge bosons) for the process at leading order, working in Feynman gauge. The external fermions are assumed to be on-shell.

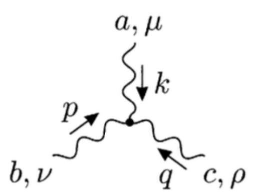
   (d) [8 marks] Show that $(k_2)_\nu \mathcal{M}^{\mu\nu}$ is non-zero in general, but vanishes when k_1 has a physical polarization.

   (e) [3 marks] Comment on the Ward identity for non-Abelian gauge theories. Is the probability to produce unphysical polarizations zero? What are the implications for loop processes?

Some useful information

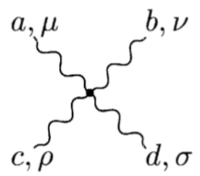
- $SU(2)$ structure constants $f^{abc} = \epsilon^{abc}$.
- $SU(2)$ generators for the fundamental representation, $t^a = \frac{1}{2}\sigma^a$, where σ^a are the Pauli matrices.
- Spinor identities: $(\not{p} - m)u(p) = 0$, and $\bar{v}(p)(\not{p} + m) = 0$.
- Feynman rules for gauge boson couplings in a non-Abelian gauge theory

3-boson vertex:



$$= g f^{abc}\big[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu\big].$$

4-boson vertex:



$$= -ig^2\big[f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})\big].$$