Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

ADVANCED FLUID DYNAMICS

TRINITY TERM 2024

Wednesday, 17th April 2024, 9:30 am to 11:30 am

You should submit answers to both questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. In this question we will examine the stability of a differentially rotating disc of material in the presence of a weak magnetic field.

(a) [5 marks] The following equation of motion

$$\rho \frac{\mathrm{D}\vec{u}}{\mathrm{D}t} = -\vec{\nabla}\left(P + \frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0}\left(\vec{B}\cdot\vec{\nabla}\right)\vec{B} - \rho\vec{\nabla}\Phi - 2\rho\vec{\Omega}\times\vec{u} - \rho\vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right),$$

describes the rate of change of the fluid velocity $\vec{u}$ in a rotating coordinate frame (with rotation rate defined by the rotation vector $\vec{\Omega}$). Discuss how this equation of motion is derived, defining all of the relevant physical quantities. Explain under what physical and mathematical assumptions this equation is valid. You may state without proof that accelerations in inertial and rotating reference frames are related by

$$\frac{\mathrm{D}\vec{u}}{\mathrm{D}t} + 2\vec{\Omega}\times\vec{u} + \vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right) = \frac{\mathrm{D}\vec{v}}{\mathrm{D}t},$$

where $\vec{u} = \mathrm{d}\vec{r}/\mathrm{d}t$ and $\vec{v}$ are the velocities in the rotating and inertial reference frames respectively, and $\mathrm{D}/\mathrm{D}t$ is a Lagrangian time derivative.

(b) [2 marks] Work in cylindrical coordinates $\vec{r} = (R, \phi, z)$, and orientate the rotational axis along $\hat{z}$, $\vec{\Omega} = \Omega\hat{z}$. For the remainder of this question we shall ignore all vertical structure of the fluid. In the limit in which pressure and magnetic forces are sub-dominant, derive the relationship between the gravitational and rotational forces in equilibrium, assuming that Ω is equal to the rate of rotation of the fluid which follows circular orbits.

(c) [3 marks] Show that a fluid element displaced from R to $R + \xi_R$, but which does not change its rate of rotation, experiences a mismatch between gravitational and rotational forces with magnitude equal to

$$|\Delta f| \simeq \rho\xi_R R\frac{\mathrm{d}\Omega^2}{\mathrm{d}R}.$$

(d) [5 marks] Consider now a more general perturbation to the fluid location, of the form $\vec{r} \to \vec{r} + \vec{\xi}$, where $\vec{\xi} = (\xi_R, \xi_\phi, 0)e^{ikz} \ll \vec{r}$. Assume that the equilibrium magnetic field is $\vec{B} = B_0\hat{z}$, and that the fluid is incompressible. Using the induction equation, or otherwise, show that the perturbation to the magnetic field caused by this fluid displacement results in a magnetic tension force equal to

$$\vec{f} = -\frac{k^2 B_0^2}{\mu_0}\vec{\xi}.$$

Give a mechanical interpretation of this force.

(e) [10 marks] Again neglecting pressure forces, show that the displaced fluid element's leading order equations of motion have solutions which are proportional to $e^{-i\omega t}$, provided that ω satisfies the dispersion relation

$$\omega^4 - \omega^2\left(2k^2v_{\mathrm{A}}^2 + 4\Omega^2 + \frac{\mathrm{d}\Omega^2}{\mathrm{d}\ln R}\right) + k^2v_{\mathrm{A}}^2\left(k^2v_{\mathrm{A}}^2 + \frac{\mathrm{d}\Omega^2}{\mathrm{d}\ln R}\right) = 0,$$

and define the quantity v_{A}. What is the stability condition for the fluid in this setup? Is this stability condition satisfied for a fluid orbiting about a central point mass, with gravitational potential $\Phi = -GM/R$? Contrast this stability criterion with that in the absence of a magnetic field.

2. This is a question about a spatially uniform suspension of bead-spring pairs in an incompressible Stokes flow with uniform velocity gradient $\nabla\mathbf{u}$. The two beads of a representative bead-spring pair are at locations $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$. They are joined by a Hookean spring with spring constant H. Each bead experiences a Stokes drag ζ when moving relative to the surrounding fluid. The mean position $\mathbf{x} = (\mathbf{r}_1 + \mathbf{r}_2)/2$ and displacement $\mathbf{R} = \mathbf{r}_2 - \mathbf{r}_1$ of the beads evolve according to

$$\dot{\mathbf{x}} = \mathbf{x} \cdot \nabla\mathbf{u}, \quad \dot{\mathbf{R}} = \mathbf{R} \cdot \nabla\mathbf{u} - \frac{2H}{\zeta}\mathbf{R} - \frac{2kT}{\zeta}\nabla_{\mathbf{R}} \log\psi, \tag{$\star$}$$

where T is the (constant) temperature of the system, and k is Boltzmann's constant. The suspension is described by the distribution function $\psi(\mathbf{R}, t)$ normalised so that $\int \psi(\mathbf{R}, t)\,\mathrm{d}\mathbf{R} = 1$. Let $\langle \cdots \rangle$ denote $\int \cdots \psi(\mathbf{R}, t)\,\mathrm{d}\mathbf{R}$.

(a) [4 marks] First consider a more general incompressible fluid system with constant density ρ and symmetric stress tensor $\boldsymbol{\sigma}$ in a space-fixed volume V with boundary ∂V and outward normal $\mathbf{n}$. The fluid velocity $\mathbf{u}$ evolves according to $\rho\partial_t\mathbf{u} = \nabla \cdot \boldsymbol{\sigma}$. Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \frac{1}{2}\rho|\mathbf{u}|^2\,\mathrm{d}V = \int_{\partial V} \mathbf{u} \cdot \boldsymbol{\sigma} \cdot \mathbf{n}\,\mathrm{d}S - \int_V \Phi\,\mathrm{d}V,$$

where $\Phi = \boldsymbol{\sigma} : \nabla\mathbf{u} = \sigma_{ij}\partial_j u_i$ in index notation. Show that $\Phi \geqslant 0$ for a Newtonian fluid.

(b) [8 marks] The distribution function $\psi(\mathbf{R}, t)$ evolves according to

$$\partial_t\psi + \nabla_{\mathbf{R}} \cdot \left(\dot{\mathbf{R}}\psi\right) = 0,$$

where $\dot{\mathbf{R}}$ is given by ($\star$) above. Show that $\mathsf{C} = \langle \mathbf{R}\mathbf{R} \rangle$ evolves according to

$$\partial_t\mathsf{C} - (\nabla\mathbf{u})^{\mathsf{T}} \cdot \mathsf{C} - \mathsf{C} \cdot (\nabla\mathbf{u}) = \frac{4kT}{\zeta}\mathsf{I} - \frac{4H}{\zeta}\mathsf{C}.$$

(c) [4 marks] Show that the viscous dissipation due to one pair of beads moving relative to the fluid around them is

$$\frac{1}{2}\zeta\left|\dot{\mathbf{R}} - \mathbf{R} \cdot \nabla\mathbf{u}\right|^2.$$

(d) [9 marks] Now suppose that the beads are also distributed in the relative velocity $\dot{\mathbf{R}}$, so the system is described by the extended distribution function

$$\Psi(\mathbf{R}, \dot{\mathbf{R}}, t) = n\left(\frac{m}{4\pi kT}\right)^{3/2} \exp\left(-\frac{m|\dot{\mathbf{R}}|^2}{4kT}\right)\psi(\mathbf{R}, t).$$

Here $\psi(\mathbf{R}, t)$ is the distribution function above, n is the (constant) number density of bead-spring pairs, and m is the mass of each bead. Let $\langle\langle \cdots \rangle\rangle$ denote $\iint \cdots \Psi(\mathbf{R}, \dot{\mathbf{R}}, t)\,\mathrm{d}\dot{\mathbf{R}}\,\mathrm{d}\mathbf{R}$. Show that

$$\left\langle\!\!\left\langle \frac{1}{2}\zeta\left|\dot{\mathbf{R}} - \mathbf{R} \cdot \nabla\mathbf{u}\right|^2 \right\rangle\!\!\right\rangle = nH\mathsf{C} : \nabla\mathbf{u},$$

and explain why this is consistent with the result in part (a).