Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

ADVANCED FLUID DYNAMICS

Trinity Term 2023

Tuesday, 18th April 2023, 9:30am - 11:30am

You should submit answers to both questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. **Magnetohydrodynamics (MHD)**

   This is a study of Alfvén waves.

   We consider an incompressible fluid of density ρ and pressure P permeated by a magnetic field $\mathbf{B}$. At equilibrium, the fluid is at rest with $\rho = \rho_0$, $P = P_0$, $\mathbf{B} = B_0\hat{\mathbf{z}}$, where ρ_0, P_0 and B_0 are uniform and $\hat{\mathbf{z}}$ is the unit vector along the z–axis. Gravity is ignored. We now consider a small perturbation such that $\rho = \rho_0 + \rho'$, $P = P_0 + P'$, $\mathbf{B} = B_0\hat{\mathbf{z}} + \mathbf{B}'$ which yields a flow velocity $\mathbf{v}$. We look for a solution as a plane wave of frequency ω and wavenumber k propagating along the z–axis.

   (a) [9 marks] Write the incompressibility condition, Navier–Stokes equation and the induction equation assuming ideal MHD. Linearise these equations and show that the wave is transverse. Establish the dispersion relation $\omega = kv_A$, and calculate the Alfvén speed v_A.

   (b) [5 marks] Calculate the magnetic and kinetic energies per unit volume associated with the wave and compare them.

   (c) [5 marks] The Poynting vector is given by $\mathbf{S} = (\mathbf{E}\times\mathbf{B})/\mu_0$. Using the expressions for $\mathbf{E}$ and $\mathbf{B}$ to first order in the perturbation, calculate $\mathbf{S}$ (retaining second order terms in the perturbation, to be consistent with the calculation of the energies per unit volume). Write the expression for $\mathbf{S}$ averaged over a period as a function of the total energy per unit volume. Explain the physical meaning of your result.

   (d) [6 marks] We now consider the effect of a small resistivity η on the propagation of an Alfvén wave with an imposed frequency ω. Resistivity contributes an additional term $\eta\nabla^2\mathbf{B}$ in the induction equation but does not affect the Navier–Stokes equation. Find the new dispersion relation. Define a dimensionless parameter which contains the resistivity and calculate the wavenumber k to first order in this parameter. Give an expression for the ratio of the damping length to the wavelength.

2. **Complex Fluids**

This question is about Stokes flow in an incompressible fluid with density ρ and dynamic viscosity μ. Consider a rigid sphere of radius a that is translating with velocity $\mathbf{U}$. The velocity and pressure fields for the flow around the sphere are

$$\mathbf{u}(\mathbf{x}) = \frac{3}{4}a\mathbf{U} \cdot \left(1 + \frac{a^2}{6}\nabla^2\right)\left(\frac{\mathsf{I}}{r} + \frac{\mathbf{x}\,\mathbf{x}}{r^3}\right), \quad p(\mathbf{x}) = \frac{3}{2}\mu a\mathbf{U} \cdot \mathbf{x}/r^3,$$

where $r = |\mathbf{x}|$ and I is the identity matrix, in a coordinate system in which the centre of the sphere is instantaneously located at $\mathbf{x} = \mathbf{0}$.

(a) [4 marks] Show that the vorticity field due to this flow is

$$\boldsymbol{\omega} = \frac{3}{2}a\mathbf{U} \times \frac{\mathbf{x}}{r^3}.$$

(b) [8 marks] By considering the stress exerted on a large spherical surface of radius much larger than a, show that the total drag force exerted on the sphere by the fluid is $-6\pi\mu a\mathbf{U}$.

(c) [4 marks] Suppose that the sphere has uniform density $\widetilde{\rho}$. Find the speed at which the sphere falls under gravity $\mathbf{g}$.

(d) [9 marks] Suppose that sphere A of radius a falls down the z-axis from far above the plane $z = 0$ to far below the plane $z = 0$. A second sphere B of radius b is held at a fixed location $(\ell, 0, 0)$ with $\ell > a + b$, but is free to rotate about its centre. Show that sphere B rotates by an angle $3a/2\ell$ in the flow due to the falling sphere A. In which direction does sphere B rotate?

Hint: the first three solid spherical harmonics are

$$\varphi^{(0)} = \frac{1}{r}, \quad \varphi_i^{(1)} = \frac{x_i}{r^3}, \quad \varphi_{ij}^{(2)} = \frac{\delta_{ij}}{r^3} - 3\frac{x_i x_j}{r^5}.$$