A19021H1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

Field Theories and Collective Phenomena in Condensed Matter

TRINITY TERM 2025

Wednesday 11th June 2:30-5:30pm

This exam paper consists of three questions. You should submit answers to all three questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. [25 marks] Consider a particle moving in one dimension with coordinate x, mass m and potential $V(x)$. Its action is

$$S[x(t')] = \int_0^t \mathrm{d}t' \left\{ \frac{m}{2} \left(\frac{\mathrm{d}x}{\mathrm{d}t'} \right)^2 - V(x(t')) \right\} . \tag{1}$$

Let H denote the quantum Hamiltonian for the particle, and let $|x_i\rangle$ and $|x_f\rangle$ denote basis states in the position representation. Then matrix elements of the time evolution operator have the path integral representation

$$\langle x_f | \mathrm{e}^{-\mathrm{i}Ht/\hbar} | x_i \rangle = \int \mathrm{D}[x(t')] \mathrm{e}^{\mathrm{i}S[x(t')]/\hbar}. \tag{2}$$

(a) [4 marks] Show (by transformation of the coordinate t' or otherwise) that matrix elements of the Boltzmann factor at inverse temperature β have the path integral representation

$$\langle x_f | \mathrm{e}^{-\beta H} | x_i \rangle = \int \mathrm{D}[x(\tau)] \mathrm{e}^{-S_{\mathrm{Euc}}[x(\tau)]/\hbar} \tag{3}$$

and give an expression for the Euclidean (or imaginary time) action $S_{\mathrm{Euc}}[x(\tau)]$.

(b) [4 marks] In the case of a free particle ($V(x) = 0$), find the path $x_0(\tau)$ for which $S_{\mathrm{Euc}}[x(\tau)]$ is stationary, as a function of the end-points x_i and x_f.

(c) [6 marks] By making the change of variables from $x(\tau)$ to $y(\tau) \equiv x(\tau) - x_0(\tau)$ in the path integral, show for the free particle that

$$\langle x_f | \mathrm{e}^{-\beta H} | x_i \rangle = \mathcal{N} \exp(-\frac{m}{2\beta\hbar^2}[x_f - x_i]^2), \tag{4}$$

Give an expression for $\mathcal{N}$ in terms of a path integral and discuss whether it depends on x_i and x_f.

(d) [5 marks] A particle with coordinate $y(t)$ undergoes a random walk as a function of time t, described by the stochastic differential equation

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \eta(t) \tag{5}$$

where $\eta(t)$ has a Gaussian distribution with mean $\langle \eta(t) \rangle = 0$ and variance $\langle \eta(t_1)\eta(t_2) \rangle = \Gamma\delta(t_1 - t_2)$. Find $\langle y(t) - y(0) \rangle$ and $\langle [y(t) - y(0)]^2 \rangle$.

(e) [6 marks] Discuss the relationship that exists (with suitable parameter values) between the paths followed by the Brownian particle and the ones that contribute to the Euclidean path integral for a free particle. Find the value of Γt required for this relationship to hold, in terms of β, m and $\hbar$.

2. [25 marks] A one-dimensional Ising antiferromagnet with nearest neighbour interactions is defined as follows. Spins $S_n = \pm 1$ are located at sites $n = 1 \ldots N$ of a chain. Periodic boundary conditions are applied, so that $S_{N+1} \equiv S_1$. The energy of a configuration is

$$H = J \sum_{n=1}^{N} S_n S_{n+1} \tag{1}$$

with exchange energy $J > 0$.

(a) [5 marks] Describe the ground states of this model and give their number: (i) for N even, and (ii) for N odd.

(b) [10 marks] Explain how the transfer matrix method may be used to study the statistical mechanics of this model. Illustrate your answer by calculating the free energy F_N and the one-point and two-point correlations functions $\langle S_1 \rangle$ and $\langle S_1 S_{n+1} \rangle$, where $\langle \ldots \rangle$ denotes an average in the canonical ensemble and $0 \leqslant n \leqslant N - 1$. You should give expressions that are exact for finite N.

(c) [5 marks] Show, in the thermodynamic limit for finite inverse temperature β, that the two-point correlation function has the form

$$\langle S_1 S_{n+1} \rangle = (-1)^n \mathrm{e}^{-n/\xi} \tag{2}$$

and give an expression for the correlation length ξ as a function of βJ.

(d) [5 marks] Calculate the low-temperature limiting form of the two-point correlation function

$$\lim_{\beta J \to \infty} \langle S_1 S_{n+1} \rangle. \tag{3}$$

for finite N. How does your result depend on whether N is even or odd? Discuss the physical reasons for the behaviour you find.

3. [25 marks] Consider a system consisting of two single-particle orbitals that may be occupied by fermions, with creation operators $a^\dagger$, $b^\dagger$ and annihilation operators a, b obeying the anticommutation relations $\{a^\dagger, a\} = \{b^\dagger, b\} = 1$ and $\{a, a\} = \{b, b\} = \{a, b\} = 0$. Number operators for the orbitals are defined by $N_a = a^\dagger a$ and $N_b = b^\dagger b$. Let $|0\rangle$ denote the vacuum for the system, which satisfies $a|0\rangle = b|0\rangle = 0$. The system has the Hamiltonian

$$H = \varepsilon(a^\dagger a + b^\dagger b) + \Delta(ab + b^\dagger a^\dagger)\,, \tag{1}$$

where ε and Δ are real, positive parameters.

(a) [6 marks] Determine the eigenvalues of N_a and N_b and write their eigenstates in terms of the creation operators and $|0\rangle$.

(b) [6 marks] Consider the Bogoliubov transformation

$$\begin{pmatrix} a \\ b^\dagger \end{pmatrix} = \begin{pmatrix} u & v \\ -v & u \end{pmatrix} \begin{pmatrix} c \\ d^\dagger \end{pmatrix}, \tag{2}$$

where u and v are real parameters. Suppose that the creation operators $c^\dagger$, $d^\dagger$ and annihilation operators c, d satisfy the full standard set of fermion anticommutation relations. Show that the transformation implies the stated anticommutation relations for $a^\dagger$, $b^\dagger$ and a, b provided that u and v satisfy a certain condition, which you should derive.

(c) [8 marks] Using this transformation, show that for a suitable choice of u and v the Hamiltonian has the form

$$H = \lambda(c^\dagger c + d^\dagger d) + \mu \tag{3}$$

and determine λ and μ as a function of ε and Δ. What are the eigenvalues of H and their degeneracies?

(d) [5 marks] Denote the vacuum for the operators $c^\dagger$ and $d^\dagger$ by $|\tilde{0}\rangle$. It obeys $c|\tilde{0}\rangle = d|\tilde{0}\rangle = 0$. Show that

$$|\tilde{0}\rangle = A\mathrm{e}^{Ba^\dagger b^\dagger}|0\rangle \tag{4}$$

where A and B are real parameters whose values you should determine as functions of ε and Δ.