A18926H1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

ANYONS AND TOPOLOGICAL QUANTUM FIELD THEORIES

Hilary Term 2025

HILARY TERM 2025

FRIDAY, 17 January 2025, 9.30 am - 11.30 am

You should submit answers to both questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

USEFUL INFORMATION (page 1/2)

Fusion multiplicity matrices: $N_{ab}^c = [N_b]_a^c \equiv N_b$ as a matrix. Associativity of fusion implies matrices commute $N_a N_b = N_b N_a$.

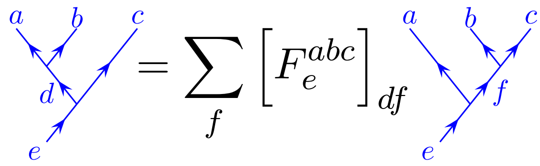
Quantum Dimensions: d_a is largest eigenvalue of N_a and $d_a d_b = \sum_c N_{ab}^c d_c$.

Total Quantum Dimension: $\mathcal{D}^2 = \sum_a d_a^2$

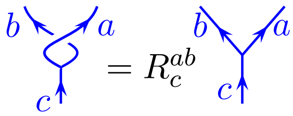
Hilbert Space Dimension: The dimension of the Hilbert space of anyons $a_1, a_2, \ldots, a_p$ on a sphere is given by $[N_{a_1} N_{a_2} \ldots N_{a_p}]_{\mathbf{1}}^{\mathbf{1}}$ where $\mathbf{1}$ is the identity or vacuum particle.

In all diagrams, indices at vertices have been suppressed. These vertex multiplicity indices can be ignored if $N_{ab}^c \in 0, 1$, which you are allowed to assume.

***F*-matrices:**



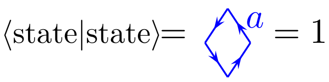
***R*-matrices:**



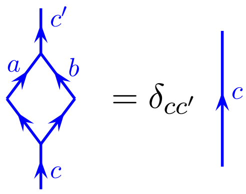
Both F and R are unitary.

Identities in Physics Normalization

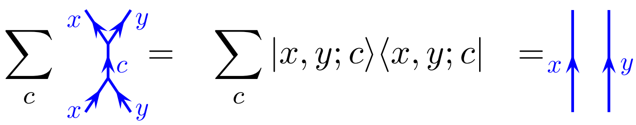
Loop Normalization:



Bubble:

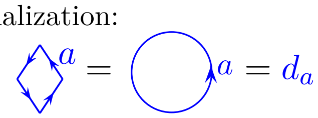


Completeness:

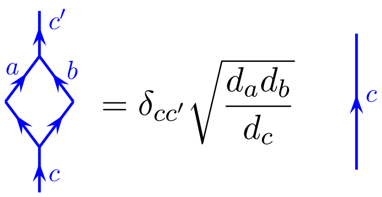


Identities in Isotopy Normalization

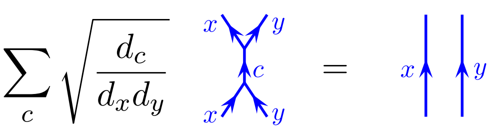
Loop Normalization:



Bubble:



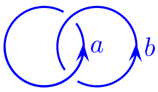
Completeness:



USEFUL INFORMATION (page 2/2)

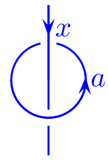
All diagrams on this page use isotopy normalization. We assume a well-behaved anyon theory with no transparent particles except the vacuum (a modular tensor category).

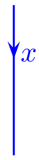
Modular S-matrix:

$S_{ab} =$ 

$\times \; \frac{1}{\mathcal{D}}$

Unlinking:



$= \frac{S_{ax}}{S_{\mathbf{1}x}}$ 

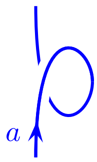
The S-matrix is unitary and symmetric. $S_{\mathbf{1}a} = d_a/\mathcal{D}$ with $\mathbf{1}$ the identity or vacuum particle. Furthermore $S_{ab} = S^*_{\bar{a}b}$ where $\bar{a}$ is the antiparticle of a.

Verlinde Formula:

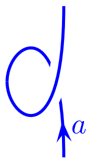
$$N^c_{ab} = \sum_x \frac{S_{ax} S_{bx} S^*_{cx}}{S_{\mathbf{1}x}}$$

with $\mathbf{1}$ the identity.

Twist θ: $\qquad T_{ab} = \theta_a \delta_{ab}$ is unitary.



$= \theta_a$ 

$=$ 

Modular Relations:
With c the chiral central charge

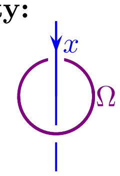
$$e^{2\pi i c/8} = \frac{1}{\mathcal{D}} \sum_a d_a^2 \theta_a \qquad \text{and} \qquad \tilde{T} = T e^{-2\pi i c/24}$$

we have the modular group relations $S^2 = C$; $C^2 = \mathbf{1}$; $(S\tilde{T})^3 = C$ with $C_{ab} = \delta_{b,\bar{a}}$.

Ω-strand:

$= \frac{1}{\mathcal{D}} \sum_a d_a$ 

Killing Property:



$= \mathcal{D} \delta_{\mathbf{1}x}$ 

TQFT Partition Function:

$$Z(S^2 \times S^1) = 1 \qquad\qquad Z(S^3) = 1/\mathcal{D}$$

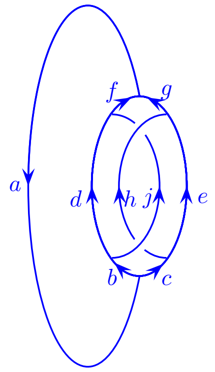
$Z(\Sigma \times S^1) =$ Degeneracy of ground state on two-dimensional closed orientable manifold Σ

Witten-Reshitikhin-Turaev:

$$Z(\mathcal{M}) = \frac{1}{\mathcal{D}} \left[e^{-2\pi i c/8} \right]^\sigma \left(\begin{array}{c} \text{Evaluate link made of } \Omega\text{-strands where} \\ \text{surgery on link in } S^3 \text{ gives } \mathcal{M} \end{array} \right)$$

where σ is the signature of the linking matrix of the link and c is the chiral central charge.

1. In this question we consider the following diagram for an anyon theory:



For simplicity, you may assume that there are no fusion multiplicities greater than 1. I.e., $N^c_{ab} = 0$ or 1 for all a, b, c.

(a) [4 marks] For which values of the variables $a, b, c, d, e, f, g, h, j$ is the diagram gauge invariant?

(b) [5 marks] A particle a is known as *transparent* if $R^{ab}_c R^{ba}_c = 1$ for all possible b and c. Explain why the set of transparent particles is closed under fusion (i.e., if a and b are transparent and $c \in a \times b$ then c is transparent also). What can you say further about the types of particles that are transparent?

(c) [4 marks] Evaluate the diagram above assuming that particle h is a transparent particle. You are allowed to assume physics normalization of the diagram.

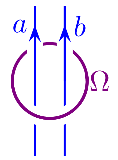
(d) [12 marks] No longer assuming h is transparent, evaluate the diagram above in terms of F-matrices and R-matrices of the theory. Again you are allowed to use physics normalization. Confirm that your result reduces to the result of part (c) above if h is transparent.

2. For this problem you should use isotopy normalization for all diagrams.

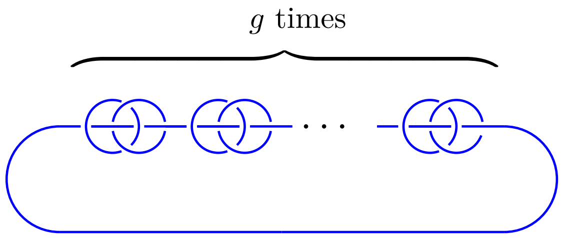
   (a) [6 marks] Consider an anyon theory with no transparent particles besides the vacuum, i.e., a modular tensor category. Let Σ_g be a g-handled torus (a closed orientable two-dimensional surface of genus g). Write the ground state degeneracy of the anyon theory on the surface Σ_g in terms of the fusion multiplicity matrices N_{ab}^c.

   (b) [6 marks] Use the Verlinde relation to simplify your result into an expression that only involves the quantum dimensions of the particles of the theory.

   (c) [3 marks] Use completeness and the killing property to simplify the following diagram, where the thick line labeled Ω is the "Kirby" or Ω-strand.



   (d) [10 marks] The following picture is a blackboard framed link diagram L



   Embedding the link L in S^3 and performing surgery on the link is known to give the three-dimensional manifold $\mathcal{M}_g = \Sigma_g \times S^1$ where Σ_g is a g-handled torus as above. Use the Witten-Reshitikhin-Turaev approach to calculate the TQFT partition function $Z(\mathcal{M}_g)$ and hence the ground state degeneracy of the anyon theory on Σ_g. Hint: Use the result of part (c). Note that the signature of the linking matrix here is $\sigma = 0$.