A15757H1

**Honour School of Mathematical and Theoretical Physics Part C**
**Master of Science in Mathematical and Theoretical Physics**

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# Advanced Quantum Theory: Path Integrals and Many-Particle Physics

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**TRINITY TERM 2025**

**Wednesday 11th June 2:30-4:30pm**

*This exam paper consists of two questions. You should submit answers to both questions.*

*You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.*

*The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.*

**Do not turn this page until you are told that you may do so**

1. [25 marks] A one-dimensional Ising antiferromagnet with nearest neighbour interactions is defined as follows. Spins $S_n = \pm 1$ are located at sites $n = 1 \ldots N$ of a chain. Periodic boundary conditions are applied, so that $S_{N+1} \equiv S_1$. The energy of a configuration is

$$H = J \sum_{n=1}^{N} S_n S_{n+1} \tag{1}$$

with exchange energy $J > 0$.

(a) [5 marks] Describe the ground states of this model and give their number: (i) for $N$ even, and (ii) for $N$ odd.

(b) [10 marks] Explain how the transfer matrix method may be used to study the statistical mechanics of this model. Illustrate your answer by calculating the free energy $F_N$ and the one-point and two-point correlations functions $\langle S_1 \rangle$ and $\langle S_1 S_{n+1} \rangle$, where $\langle \ldots \rangle$ denotes an average in the canonical ensemble and $0 \leqslant n \leqslant N-1$. You should give expressions that are exact for finite $N$.

(c) [5 marks] Show, in the thermodynamic limit for finite inverse temperature $\beta$, that the two-point correlation function has the form

$$\langle S_1 S_{n+1} \rangle = (-1)^n \mathrm{e}^{-n/\xi} \tag{2}$$

and give an expression for the correlation length $\xi$ as a function of $\beta J$.

(d) [5 marks] Calculate the low-temperature limiting form of the two-point correlation function

$$\lim_{\beta J \to \infty} \langle S_1 S_{n+1} \rangle. \tag{3}$$

for finite $N$. How does your result depend on whether $N$ is even or odd? Discuss the physical reasons for the behaviour you find.

2. [25 marks] Consider a system consisting of two single-particle orbitals that may be occupied by fermions, with creation operators $a^\dagger$, $b^\dagger$ and annihilation operators $a$, $b$ obeying the anticommutation relations $\{a^\dagger, a\} = \{b^\dagger, b\} = 1$ and $\{a, a\} = \{b, b\} = \{a, b\} = 0$. Number operators for the orbitals are defined by $N_a = a^\dagger a$ and $N_b = b^\dagger b$. Let $|0\rangle$ denote the vacuum for the system, which satisfies $a|0\rangle = b|0\rangle = 0$. The system has the Hamiltonian

$$H = \varepsilon(a^\dagger a + b^\dagger b) + \Delta(ab + b^\dagger a^\dagger)\,, \tag{1}$$

where $\varepsilon$ and $\Delta$ are real, positive parameters.

(a) [6 marks] Determine the eigenvalues of $N_a$ and $N_b$ and write their eigenstates in terms of the creation operators and $|0\rangle$.

(b) [6 marks] Consider the Bogoliubov transformation

$$\begin{pmatrix} a \\ b^\dagger \end{pmatrix} = \begin{pmatrix} u & v \\ -v & u \end{pmatrix} \begin{pmatrix} c \\ d^\dagger \end{pmatrix}, \tag{2}$$

where $u$ and $v$ are real parameters. Suppose that the creation operators $c^\dagger$, $d^\dagger$ and annihilation operators $c$, $d$ satisfy the full standard set of fermion anticommutation relations. Show that the transformation implies the stated anticommutation relations for $a^\dagger$, $b^\dagger$ and $a$, $b$ provided that $u$ and $v$ satisfy a certain condition, which you should derive.

(c) [8 marks] Using this transformation, show that for a suitable choice of $u$ and $v$ the Hamiltonian has the form

$$H = \lambda(c^\dagger c + d^\dagger d) + \mu \tag{3}$$

and determine $\lambda$ and $\mu$ as a function of $\varepsilon$ and $\Delta$. What are the eigenvalues of $H$ and their degeneracies?

(d) [5 marks] Denote the vacuum for the operators $c^\dagger$ and $d^\dagger$ by $|\tilde{0}\rangle$. It obeys $c|\tilde{0}\rangle = d|\tilde{0}\rangle = 0$. Show that

$$|\tilde{0}\rangle = A\mathrm{e}^{Ba^\dagger b^\dagger}|0\rangle \tag{4}$$

where $A$ and $B$ are real parameters whose values you should determine as functions of $\varepsilon$ and $\Delta$.