A15278S1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

COLLISIONLESS PLASMA PHYSICS

Trinity Term 2025

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20 JUNE 2025, NOON

You should submit answers to all three questions.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

1. Consider a plasma composed of ions, with mass m_i and charge q_i, and electrons, with mass m_e, immersed in a uniform equilibrium magnetic field $\boldsymbol{B}_0 = B_0\hat{\boldsymbol{z}}$, where (x, y, z) are the usual Cartesian coordinates. All other symbols have their usual definitions.

   A wave with frequency $\omega \sim \omega_{\rm pe} \sim \Omega_e \gg k v_{\rm th}$ is launched into the plasma.

   (a) [5 marks] Using the linearised cold-plasma equations of motion for ions and electrons, derive the conductivity tensor σ for each species. Show that the dielectric tensor $\boldsymbol{\epsilon}$ for the plasma is

   $$\boldsymbol{\epsilon} = \begin{pmatrix} \epsilon_\perp & \mathrm{i}g & 0 \\ -\mathrm{i}g & \epsilon_\perp & 0 \\ 0 & 0 & \epsilon_\parallel \end{pmatrix},$$

   where $\epsilon_\perp$, $\epsilon_\parallel$, and g depend on $\omega_{\rm pe}^2$, Ω_e, and ω.

   (b) [5 marks] We now choose $\boldsymbol{k}$ to lie in the (x, z) plane, making an angle θ with the background field $\boldsymbol{B}_0 = B_0\hat{\boldsymbol{z}}$. That is, $\boldsymbol{k} = k\sin\theta\,\hat{\boldsymbol{x}} + k\cos\theta\,\hat{\boldsymbol{z}}$, where $k = |\boldsymbol{k}|$. Show that the waves satisfy

   $$\begin{pmatrix} \epsilon_\perp - n^2\cos^2\theta & \mathrm{i}g & n^2\cos\theta\sin\theta \\ -\mathrm{i}g & \epsilon_\perp - n^2 & 0 \\ n^2\cos\theta\sin\theta & 0 & \epsilon_\parallel - n^2\sin^2\theta \end{pmatrix} \delta\boldsymbol{E} = 0,$$

   where the index of refraction is given by $n = kc/\omega$ (where c is the speed of light).

   Show that the cold plasma dispersion relation can be written as

   $$\tan^2\theta = -\frac{\epsilon_\parallel(n^2 - R)(n^2 - L)}{(\epsilon_\perp n^2 - RL)(n^2 - \epsilon_\parallel)}, \qquad (1)$$

   where R and L are functions of $\epsilon_\perp$, $\epsilon_\parallel$, and g that you should determine.

   (c) [3 marks] Using equation (1), write down a general condition for a resonance and a general condition for a cutoff.

   (d) [2 marks] Explain how the following special cases can be recovered from the dispersion relation (1): plasma oscillations, waves with right-handed polarisation, waves with left-handed polarisation, ordinary waves (O-modes), and extra-ordinary waves (X-modes).

   (e) [3 marks] Show that electrostatic waves satisfy the dispersion relation

   $$k_\perp^2\epsilon_\perp + k_\parallel^2\epsilon_\parallel = 0.$$

   Explain how, for parallel propagation, this dispersion relation is related to the resonance condition you derived in part (c).

   (f) [2 marks] Show that, for the ordinary wave (O-wave), the electric field only moves particles along $\boldsymbol{B}_0$. Is the O-wave electrostatic? Explain your answer clearly.

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2. An O-mode (as outlined above) is launched into a dense plasma column where the plasma frequency varies with position according to

   $$\omega_{\rm pe}^2(x) = \omega_{\rm pe0}^2\left[1 - \left(\frac{x}{L}\right)^2\right], \qquad |x| \leqslant L.$$

   The plasma density is highest at $x = 0$ (with $\omega_{\rm pe}(0) = \omega_{\rm pe0}$) and decreases towards the edges (with $\omega_{\rm pe}^2(\pm L) = 0$). Assume that the wave frequency is chosen such that $\omega < \omega_{\rm pe0}$, so that the wave is evanescent in the central high-density region (for $|x| < x_T$), but propagates in the outer, low-density wings (for $|x| > x_T$). Here, x_T denotes the turning point, i.e., the location where the wave transitions between evanescent and propagating behaviour. In this question, any limits of special functions that are proved in the appendices of the printed lecture notes may be used in your solution without proof, provided that the result is clearly stated and referenced.

(a) [5 marks] What is the WKB approximation for a wave of characteristic wavelength λ propagating through this plasma? Define the eikonal function S, the spatially varying wavenumber $\boldsymbol{k}$, and carefully state all ordering assumptions. Show that, to lowest order in the appropriate expansion parameter, the frequency ω and wavenumber k satisfy the cold-plasma dispersion relation at every point in space. That is, show that a WKB solution for the perturbed electric field $\delta E(x)$ obeys the wave equation

$$\frac{\partial^2 \delta E}{\partial x^2} + k^2(x)\, \delta E = 0,$$

with the local dispersion relation

$$k^2(x) = \frac{\omega^2}{c^2}\left[1 - \frac{\omega_{\mathrm{pe}}^2(x)}{\omega^2}\right].$$

(b) [2 marks] Show that the turning points, $x = \pm x_T$ occur at $x_T = L\sqrt{1 - \omega^2/\omega_{\mathrm{pe0}}^2}$.

(c) [3 marks] Write down the WKB solutions for $\delta E(x)$ in each of the three regions: Region I ($x < -x_T$), Region II ($-x_T < x < x_T$), and Region III ($x > x_T$). Carefully define the local wavenumber in each region. Ensure that you include all appropriate phase integrals and normalisation factors.

(d) [5 marks] Near a turning point (say, at $x = -x_T$) the wave equation can be linearised. Show that the wave equation can be reduced to the standard Airy equation by introducing a change of variables.

(e) [8 marks] Derive the connection formulas that relate the WKB solution in Region I to the Airy-function solution near $x = -x_T$. In particular, show that the cosine component in the WKB region — with the $-\pi/4$ phase shift that arises from matching to the Airy function — corresponds to the decaying exponential in the evanescent region. You may use the asymptotic forms of the Airy function provided in the lecture notes.

(f) [8 marks] Propagate the solution through Region II from $x = -x_T$ to $x = x_T$. We define the tunnelling "action" by

$$\eta = \int_{-x_T}^{x_T} k_2(x')\, dx'.$$

Derive the appropriate connection formulas at $x = x_T$ that match the evanescent solution to the WKB solution in Region III. By imposing an appropriate restriction on the WKB solution in Region III, obtain a relationship between the coefficients and show that the transmitted amplitude is proportional to $e^{-\eta}$. Conclude that the transmission coefficient is, to leading order,

$$T \simeq e^{-2\eta}.$$

(g) [4 marks] Let R be the reflection coefficient. What is the value of $R + T$? Give a reason for your answer. How would this value change if one of the cutoffs was replaced by a resonance.

(h) [5 marks] Finally, evaluate η explicitly. Discuss briefly the physical significance of this result.

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3. **Interchange Instability in KMHD.** Consider a magnetised plasma in the Kinetic MHD approximation, consisting of any number of particle species α, each with its own mass m_α, charge q_α, number density n_α, pressure p_α, and temperature $T_\alpha = p_\alpha/n_\alpha$. Assume that the plasma is in a z-pinch equilibrium, i.e., that the equilibrium magnetic field $\boldsymbol{B}_0 = B_0\boldsymbol{b}_0$ is purely circular while its strength $B_0 = |\boldsymbol{B}_0|$ and the equilibrium profiles of density and pressure vary only in the radial direction. We shall work in local coordinates (x, y, z) such that their corresponding unit vectors are: $\hat{\boldsymbol{z}} = \boldsymbol{b}_0$ in the azimuthal direction, $\hat{\boldsymbol{x}}$ in the radial direction, and $\hat{\boldsymbol{y}} = \boldsymbol{b}_0 \times \hat{\boldsymbol{x}}$. In this notation, the relationships between the local values of the equilibrium profiles, the field's direction, and their gradients in a z-pinch equilibrium are

$$\sum_\alpha q_\alpha n_{0\alpha} = 0, \quad \boldsymbol{\nabla}\cdot\boldsymbol{b}_0 = 0, \quad \boldsymbol{b}_0\cdot\boldsymbol{\nabla}\boldsymbol{b}_0 = -\frac{\hat{\boldsymbol{x}}}{R}, \quad \boldsymbol{b}_0\cdot\boldsymbol{\nabla}\hat{\boldsymbol{x}} = \frac{\boldsymbol{b}_0}{R}, \quad \frac{1}{R} - \frac{1}{L_B} = \frac{\beta}{2L_p},$$

where R is the radius of curvature of the magnetic field, $\beta = 8\pi p_0/B_0^2$, $p_0 = \sum_\alpha p_{0\alpha}$, and $L_B^{-1} = -\mathrm{d}\ln B_0/\mathrm{d}x$ and $L_p^{-1} = -\mathrm{d}\ln p_0/\mathrm{d}x$ are the local gradient scale lengths of the magnetic and thermal pressure, respectively; there is no equilibrium parallel electric field or mean motion of any kind. For simplicity, we shall take the pressure gradient to be much steeper than any of the magnetic field's gradients, but also $\beta \ll 1$, so that

$$\frac{1}{L_p} \gg \frac{1}{L_B} \approx \frac{1}{R} \gg \frac{\beta}{L_p}. \tag{2}$$

We shall assume the equilibrium distribution function to be

$$f_{0\alpha}(x, \mu, v_\parallel) = \frac{n_{0\alpha}}{\pi^{3/2} v_{\mathrm{th}\alpha}^3}\, e^{-(2\mu B_0 + v_\parallel^2)/v_{\mathrm{th}\alpha}^2},$$

where μ is the first adiabatic invariant, $v_\parallel$ the parallel velocity, and $v_{\mathrm{th}\alpha} = \sqrt{2T_{0\alpha}/m}$ the thermal speed of the particles of species α. In what follows, we shall consider collisionless, infinitesimal perturbations of this equilibrium: $f_\alpha = f_{0\alpha} + \delta f_\alpha$, $B = B_0 + \delta B$, $\boldsymbol{b} = \boldsymbol{b}_0 + \delta\boldsymbol{b}$, mean velocities $\boldsymbol{u}$, and parallel electric field $E_\parallel$.

(a) [5 marks] Show that the equilibrium described above is indeed a steady-state solution of the Kinetic MHD system of equations.

(b) [12 marks] Linearise the KMHD equations for $\boldsymbol{B}$ and $\boldsymbol{u} = \partial\boldsymbol{\xi}/\partial t$ (where $\boldsymbol{\xi}$ is the displacement vector) around this equilibrium, assume that perturbations vary on scales much shorter than the equilibrium so all equilibrium quantities and their gradients can be treated as locally uniform, Fourier-transform the perturbations in time ($\partial_t \to -i\omega$) and in the two spatial directions transverse to the pressure gradient ($\hat{\boldsymbol{y}}\cdot\boldsymbol{\nabla} \to ik_y$, $\boldsymbol{b}_0\cdot\boldsymbol{\nabla} \to ik_\parallel$). Consider perturbations that have no variation in the x direction at all, but $k_y \gg k_\parallel \sim \omega/v_{\mathrm{A}}$. Show that such perturbations will be pressure balanced, viz.,

$$\frac{\delta B}{B_0} = -\frac{\beta}{2}\frac{\delta p_\perp}{p_0}, \tag{3}$$

and satisfy

$$\left(\omega^2 - k_\parallel^2 v_{\mathrm{A}}^2\right)\xi_x = -\frac{1}{R}\frac{\delta p_\perp + \delta p_\parallel}{\rho_0},$$

where $\delta p_\perp$ and $\delta p_\parallel$ are, respectively, the perpendicular and parallel pressure perturbations (defined as the sums of the corresponding pressure perturbations associated with all species), $\rho_0 = \sum_\alpha m_\alpha n_{0\alpha}$, and $v_{\mathrm{A}} = B_0/\sqrt{4\pi\rho_0}$ is the Alfvén speed.

(c) [12 marks] Use the KMHD drift-kinetic equation to calculate the pressure perturbations and hence derive the dispersion relation

$$\omega^2 = k_\parallel^2 v_{\mathrm{A}}^2 - \frac{2p_0}{\rho_0 R L_p}. \tag{4}$$

In doing this calculation, you may assume that, in view of the pressure balance (3) and $\beta \ll 1$, $\delta B/B_0$ can always be neglected in comparison with $\delta p_\perp/p_0$ and, as per (2), that the magnetic field's gradients can be neglected in comparison with the pressure gradient.

(d) [3 marks] If you have done correctly the calculation leading to (4), you must have paid careful attention to the fact that the variables and the parallel gradients in the KMHD kinetic equation are defined with respect to the exact magnetic field, which has to be perturbed correctly when the equation is linearised. You should have beeen able to prove in the limits (2) that, remarkably simply,

$$\delta f_\alpha \approx -\xi_x \frac{\partial f_{0\alpha}}{\partial x}.$$

Explain what this means physically and why it is obvious that it should be so.

(e) [4 marks] What is the physical nature of the perturbations described by the dispersion relation (4) and how does it depend on their wavenumber? Describe their time evolution and the physical reasons for it. Would you expect the dispersion relation to be different if, instead of KMHD, the MHD approximation were used? Make an educated guess as to what new effects might appear if the assumptions (2) are relaxed.

(f) [4 marks] What is the difference between the physics that you have derived here and the physics of the ion-temperature-gradient (ITG) instability? Why have you not simply found the plasma to be ITG-unstable, and in what limit, different from the one considered here, would you expect this to happen?