A15276H1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

Cosmology

TRINITY TERM 2025

Thursday 5th June 2:30-4:30pm

This exam paper consists of three questions. You should submit answers to two out of three questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. (a) [5 marks] Consider a universe filled with only non-relativistic matter, but with a total matter density that is a fraction $\Omega < 1$ smaller than the critical density. The dynamics of the scale factor $a(t)$ as a function of cosmic time t are governed by the equation:

$$H(a) \equiv \frac{\dot{a}}{a} = H_0\sqrt{\Omega a^{-3} + (1-\Omega)a^{-2}}, \tag{1}$$

where H_0 is the expansion rate today. What principles and physical considerations lead us to this equation? What components should dominate the Universe's expansion at early and late times? What should the behaviour of $a(t)$ be in those regimes?

(b) [6 marks] Solve Eq. 1 to find the following implicit solution:

$$t = \frac{\Omega}{2H_0(1-\Omega)^{3/2}}[\sinh\theta - \theta], \quad a = \frac{\Omega}{2(1-\Omega)}[\cosh\theta - 1].$$

Show that this solution recovers the asymptotic behaviour at early and late times you guessed in the previous part.
[Hint: you may find the following changes of variables useful

$$x \equiv \frac{1-\Omega}{\Omega}a, \quad z = 2x = \cosh\theta - 1$$

when solving the differential equation.]

(c) [6 marks] The linear growth equation for the matter overdensity δ in the Newtonian limit reads:

$$\frac{d}{da}\left(a^3 H(a)\frac{d\delta}{da}\right) = \frac{3}{2}H(a)\Omega_m(a)\,a\,\delta, \tag{2}$$

where $\Omega_m(a)$ is the fractional matter density when the scale factor takes the value a. Change variables to x as defined in Eq. 2, and find the following simpler version of this equation:

$$\delta'' + \frac{1}{2}\left(\frac{3}{x} + \frac{1}{1+x}\right)\delta' - \frac{3}{2}\frac{1}{x^2(1+x)}\delta = 0,$$

where $\delta' \equiv d\delta/dx$. Show that

$$\delta_1(x) \equiv \sqrt{(1+x)/x^3} \tag{3}$$

is a solution to this equation. Is this a growing or decaying mode?

(d) [5 marks] Using Eq. 3, or otherwise, find the second solution:

$$\delta_2(x) = \sqrt{\frac{1+x}{x^3}}\int_0^x dy\left(\frac{y}{1+y}\right)^{3/2}. \tag{4}$$

Try to derive the solution itself, rather than simply substituting it in the equation, to get full marks.
Without solving this integral, show that, at early times ($x \ll 1$), $\delta_2 \propto x$, while at late times ($x \gg 1$) δ_2 tends to a constant. Why does this behaviour make sense?

(e) [3 marks] Solve the integral in Eq. 4 and find a closed form for $\delta_2(x)$.
[Hint: The change of variables suggested in part b) above may come in handy.]

2. (a) [5 marks] At linear order, and in the Newtonian limit, the equation describing the evolution of the peculiar velocity field $\mathbf{v}$ is given by

$$\dot{\mathbf{v}} + H\mathbf{v} + \tfrac{c_s^2}{a}\nabla\delta + \tfrac{1}{a}\nabla\psi = 0.$$

Here, a is the scale factor, $H \equiv \frac{\dot{a}}{a}$ is the expansion rate, δ is the matter overdensity, c_s is the sound speed and ψ is the Newtonian potential. Define the "transverse" or "vortical" component of the velocity field, $\mathbf{v}_\perp$, and show that its linear evolution is governed by the equation

$$\dot{\mathbf{v}}_\perp + H\mathbf{v}_\perp = 0.$$

Solving this equation, show that this component will decay as $\mathbf{v}_\perp \propto 1/a$.

(b) [8 marks] Consider a perturbed flat FRW metric with only vector modes switched on. The metric, using conformal time, then reads

$$d\tau^2 = a^2\left[d\eta^2 - (\delta_{ij} - J_{ij})dx^i\,dx^j\right],$$

with $J_{ij} \equiv \partial_j F_i + \partial_i F_j$, where $\mathbf{F}$ is purely transverse ($\partial_i F_i = 0$). The affine connection coefficients at linear order in $\mathbf{F}$ for this metric are given by:

$$\Gamma^0_{00} = \mathcal{H}, \quad \Gamma^0_{ij} = \mathcal{H}(\delta_{ij} - J_{ij}) - \frac{1}{2}J'_{ij}, \quad \Gamma^i_{0j} = \mathcal{H}\delta_{ij} - \frac{1}{2}J'_{ij}, \quad \Gamma^i_{jk} = -\gamma^i_{jk},$$

with $\gamma^i_{jk} \equiv \frac{1}{2}(\partial_j J_{ik} + \partial_k J_{ij} - \partial_i J_{jk})$, $\mathcal{H} \equiv a'/a$, and primes denote derivatives with respect to η (e.g. $J'_{ij} \equiv \partial J_{ij}/\partial\eta$).

Derive the first two coefficients (Γ^0_{00}, and Γ^0_{ij}), and show that Γ^0_{0i} is zero. Prove also that

$$\gamma^i_{ij} = 0.$$

(c) [6 marks] For a universe filled with a perfect fluid, considering only vector velocity modes, the energy-momentum tensor at linear order reads:

$$T^0_0 = \bar{\rho}, \quad T^i_j = -\bar{P}\delta_{ij}, \quad T^i_0 = -T^0_i = (\bar{\rho} + \bar{P})v^i_\perp,$$

where $\bar{\rho}$ and $\bar{P}$ are the mean background density and pressure. Combine this with the affine connection elements from the previous question to derive the energy-momentum conservation equations in terms of $v^i_\perp$. When doing this, remember that, since we are dealing with vector modes, $\partial_i v^i_\perp = 0$.

Hint: remember that the energy-momentum conservation equations read

$$\nabla_\mu T^\mu_\nu \equiv \partial_\mu T^\mu_\nu + \Gamma^\mu_{\mu\sigma}T^\sigma_\nu - \Gamma^\sigma_{\mu\nu}T^\mu_\sigma = 0.$$

(d) [6 marks] The equation for $\nu = 0$ only receives a background contribution. What is its physical interpretation? The equation for $\nu = i$, on the other hand, is purely perturbative. Show that it reduces to

$$q'_i + 4\mathcal{H}q_i = 0,$$

where $\mathbf{q} \equiv (\bar{\rho} + \bar{P})\mathbf{v}_\perp$. Solve this to find that $\mathbf{q} \propto a^{-4}$. What does this imply for $\mathbf{v}_\perp$? How does it compare with the result you found for Newtonian perturbations?

3. (a) [6 marks] Describe the physical mechanism that leads to the emission of the Cosmic Microwave Background (CMB). Why is the CMB emitted at redshift $z \simeq 1100$, and not earlier?

Calculate the angle subtended on the sky by the particle horizon at the time of recombination. A rough estimate is more than enough: for now assume that the Universe is matter-dominated throughout its history. Why is the value you obtained problematic? How can inflation solve this problem?

(b) [6 marks] The fluctuations in the temperature of the CMB in a given direction $\hat{\mathbf{n}}$ are given by

$$\frac{\delta T(\hat{\mathbf{n}})}{T} = \left[\frac{\delta_\gamma}{4} + \psi - \hat{\mathbf{n}} \cdot \mathbf{v}\right]_{\rm rec} + 2\int_0^{\chi_{\rm LSS}} d\chi\, \psi'(\chi\hat{\mathbf{n}}, z(\chi)), \qquad (1)$$

where δ_γ is the relative perturbation in the energy density of photons, ψ is the Newtonian gravitational potential, $\mathbf{v}$ is the peculiar velocity field, and $\chi_{\rm LSS}$ is the distance to the last-scattering surface. The quantities in brackets are evaluated at the time of recombination, and in the direction of $\hat{\mathbf{n}}$. ψ' is the partial derivative of ψ with respect to conformal time. Explain the physical origin of the 4 terms present in this equation, and why they contribute to the total temperature fluctuation. Explain why the last term is the only contribution you need to take into account when correlating the CMB temperature fluctuations with the overdensity of galaxies at low redshifts.

(c) [8 marks] The cross-power spectrum, in the Limber approximation, between two fields, $u(\hat{\mathbf{n}})$ and $v(\hat{\mathbf{n}})$, defined on the sphere is defined to be

$$C_\ell^{uv} = \int \frac{d\chi}{\chi^2} q_u(\chi) q_v(\chi)\, P_{UV}\left(\frac{\ell + 1/2}{\chi}, z(\chi)\right), \qquad (2)$$

where q_u and q_v are the radial kernels associated with each field, U and V are the 3-dimensional physical quantities they correspond to, and $P_{UV}(k, z)$ is the 3D power spectrum of the latter. Our notation assumes that the projected and 3D quantities are related via

$$u(\hat{\mathbf{n}}) = \int d\chi\, q_u(\chi)\, U(\chi\hat{\mathbf{n}}, z(\chi)). \qquad (3)$$

Show that the cross-power spectrum between the CMB temperature fluctuations and the overdensity of a sample of galaxies is given by

$$C_\ell^{\delta_g\, \delta T/T} = \frac{3H_0^2\Omega_m}{(\ell + 1/2)^2} \int d\chi\, \left\{H^2(z)\, [1 - f(z)]\, b_g\, p(z)\right\}_{z=z(\chi)}\, P\left(\frac{\ell + 1/2}{\chi}, z(\chi)\right). \qquad (4)$$

where $P(k, z)$ is the matter power spectrum, $H(z)$ is the expansion rate, $p(z)$ is the redshift distribution of the galaxy sample, b_g is the galaxy bias, and $f(z) \equiv d\log\delta/d\log a$ is the growth rate.

[Hint: you may find the Poisson equation, $\nabla^2\psi = 4\pi G a^2 \bar{\rho}\delta$ (where $\bar{\rho}$ is the background energy density in matter), handy in order to relate everything to the matter overdensity.]

(d) [5 marks] Consider two samples of galaxies, one concentrated around $z \simeq 0.3$ and one at $z \simeq 4$. Assuming we live in a flat ΛCDM universe with $\Omega_m \simeq 0.3$, which of them would you expect to exhibit a stronger correlation with the CMB anisotropies?