A15272H1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

Advanced Quantum Field Theory for Particle Physics

Trinity Term 2025

Wednesday 23rd April 9:30am-12:30pm

This exam paper consists of three questions, each marked out of 25. You should submit answers to all three questions. You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question

Do not turn this page until you are told that you may do so

1. Consider a scalar field ϕ in four dimensions. The Lagrangian, including a quartic interaction is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \,. \tag{1}$$

(a) [2 marks] Write down the path integral expression for the partition function for this theory.

(b) [2 marks] Find the equation of motion for the Lagrangian $\mathcal{L}$ in equation (1).

(c) [5 marks] What are renormalizable terms in simple power counting? What is the justification of not writing down the non-renormalizable terms in the Lagrangian? Does the Lagrangian contain all possible renormalizable terms? Explain why any missing renormalizable terms can be consistently omitted.

(d) [4 marks] Write down all the possible terms of dimension-6 in the scalar field theory. Ignoring total derivative operators, you should find 7 operators.

(e) [4 marks] Using integration-by-parts, show that you can reduce the set of operators to the set $\{\phi^6, \phi^3\Box\phi, (\Box\phi)^2\}$, where $\Box \equiv \partial_\mu\partial^\mu$.

(f) [4 marks] Certain higher-dimensional operators can also be removed (at that order in power counting) by using equations of motion, or more precisely by a field redefinition. Consider adding an operator to the Lagrangian of the form

$$\mathcal{O}(x) = \epsilon f(\phi(x))E(\phi(x)) \,,$$

where $f(\phi)$ is some local function of ϕ and $E(\phi) = 0$ is the equation of motion. Show that this operator can be removed at $O(\epsilon)$ by a field redefinition $\phi \to \phi - \epsilon f(\phi)$ in the partition function. (You can neglect the contribution from the Jacobian arising from the field redefinition.)

(g) [4 marks] Using the equation of motion found in part (b) above, reduce further the set of dimension-6 operators.

2. Consider QED in four dimensions, using Feynman gauge throughout.

   (a) [5 marks] Draw a labeled one loop diagram for the 1PI correction to the photon propagator, $i\Pi_{\mu\nu}(q^2)$. Write down the Feynman amplitude for $\Pi_{\mu\nu}(q^2)$. You do not have to simplify the expression yet.

   (b) [5 marks] Using the Ward identity the 1PI correction can be put in the form

$$\Pi_{\mu\nu}(q^2) = (q^2 g_{\mu\nu} - q_\mu q_\nu)\Pi(q^2)\,.$$

   Show that in $d = 4 - 2\epsilon$,

$$\Pi(q^2) = \frac{4ie^2}{(d-1)q^2}\int \frac{d^d\ell}{(2\pi)^d}\frac{-(d-2)\ell\cdot(\ell+q) + dm^2}{((\ell+q)^2 - m^2)(\ell^2 - m^2)}\,.$$

   Extract the divergent part of $\Pi(q^2)$ as $\epsilon \to 0$. (Hint: you may find it easier to massage the expression to scalar integrals provided below rather than use Feynman parametrization.)

   (c) [5 marks] Draw the one-loop diagram for the 1PI correction to the fermion propagator,

$$-i\Sigma(p) \equiv -i(\not{p}S(p^2) + M(p^2))\,,$$

   where $S(p^2)$ and $M(p^2)$ are Lorentz scalars. Write down the amplitude for $\Sigma(p)$. Using $\mathrm{tr}(\not{p}\Sigma(p))$, extract the divergent part of $S(p^2)$.

   (d) [5 marks] Draw the one-loop diagram for the correction to the fermion-photon vertex,

$$-ie\Gamma^\mu(q^2) \equiv -ie\left(\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)\right)\,,$$

   where q^μ is the momentum on the photon line, $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, and $F_1(q^2)$ and $F_2(q^2)$ are Lorentz scalars. Write down the amplitude. Using $\mathrm{tr}(\Gamma^\mu\gamma_\mu)$, extract the divergent part of $F_1(q^2)$.

   (e) [5 marks] What is the $\overline{\mathrm{MS}}$ renormalization scheme? Using results from above, calculate the counterterms δ_1, δ_2 and δ_3 at one loop in the $\overline{\mathrm{MS}}$ scheme, and show that $\delta_1 = \delta_2$.

Some useful information:

- Scalar integrals ($d = 4 - 2\epsilon$)

$$\mathcal{I}_2(p^2) = \int \frac{d^d\ell}{(2\pi)^d}\frac{1}{((\ell+p)^2 - m^2)(\ell^2 - m^2)} = \frac{i}{16\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right)$$

$$\mathcal{I}_1 = \int \frac{d^d\ell}{(2\pi)^d}\frac{1}{\ell^2 - m^2} = \frac{im^2}{16\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right)$$

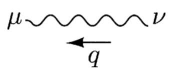
- Gamma matrix trace identities:

$$\mathrm{tr}(1) = 4$$
$$\mathrm{tr}(\text{any odd \# of } \gamma's) = 0$$
$$\mathrm{tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$$

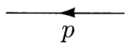
- Gamma matrix contraction identities:

$$\gamma^\mu \gamma_\mu = d$$
$$\gamma_\mu \gamma^\nu \gamma^\mu = -(d-2)\gamma^\nu$$
$$\gamma_\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\mu = -2\gamma^\sigma \gamma^\rho \gamma^\nu + (4-d)\gamma^\nu \gamma^\rho \gamma^\sigma$$

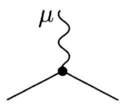
- Feynman rules for QED in Feynman gauge:



$$= \frac{-ig_{\mu\nu}}{q^2 + i\epsilon} \qquad \text{(Feynman gauge)}$$

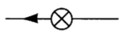


$$= \frac{i}{\not{p} - m + i\epsilon}$$

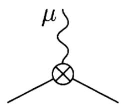


$$= -ie\gamma^\mu$$

$$= -i(g^{\mu\nu}q^2 - q^\mu q^\nu)\delta_3$$



$$= i(\not{p}\delta_2 - \delta_m)$$



$$= -ie\gamma^\mu \delta_1$$

3. The Lagrangian for Abelian gauge theory is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \, .$$

(a) [5 marks] Follow the Faddeev-Popov procedure to derive the gauge-fixing and ghost terms in the partition function. Choose the generalized Lorenz gauge, $\partial_\mu A^\mu - \omega(x) = 0$ as the gauge fixing condition. Show that the ghost terms decouple in this case. Do they also decouple for non-Abelian gauge fields?

(b) [5 marks] The Lagrangian for the Abelian Higgs model is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + |D_\mu \phi|^2 - \frac{1}{2}\lambda(\phi^\dagger \phi - \frac{1}{2} f^2)^2 \, .$$

Write down the Lagrangian in terms of field fluctuations around the classical vacuum of the theory $h(x)$ and $\pi(x)$ defined by

$$\phi(x) = \frac{1}{\sqrt{2}}(f + h(x)) e^{i\pi(x)/f} \, .$$

Show that the Lagrangian is invariant under a local symmetry under which $\pi(x)$ also transforms as

$$\pi(x) \to \pi(x) + f\alpha(x) \, . \qquad (1)$$

Why is this a well-motivated form of parametrizing the fluctuations?

(c) [5 marks] Define the Unitarity gauge. What is the form of the gauge field propagator in the Unitarity gauge? Discuss the high energy behaviour of the propagator and its relation to power counting.

(d) [5 marks] The R_ξ gauge condition is $G(A) = 0$, with

$$G(A) = \frac{1}{\sqrt{\xi}} \left(\partial_\mu A^\mu - \xi e f \left(1 + \frac{h(x)}{f}\right)^2 \pi(x) \right) .$$

Explain the advantage of using this gauge fixing condition, and using equation (1) derive the ghost terms in the Lagrangian. Do the ghost terms decouple?

(e) [5 marks] Write down the form of the gauge boson propagator in the R_ξ gauge. In a sentence or two, discuss how this shows that spontaneously broken gauge theories are renormalizable in the power counting sense.