Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

ADVANCED FLUID DYNAMICS

Trinity Term 2025

Tuesday 22nd April, 09:30 - 11.30am

You should submit answers to both questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. **Magnetohydrodynamics**

In this question we shall examine the effects that a finite resistivity has on the evolution of the magnetic field in an electrically conducting fluid.

(a) [10 marks] In the rest frame of the fluid, the electric field $\vec{E}'$ is related to the current density $\vec{J}$ by

$$\vec{E}' = \eta\vec{J},$$

where η is a constant known as the *Ohmic resistivity*. During the course we assumed $\eta = 0$.

By performing a Lorentz transform from a lab frame with electric field $\vec{E}$ and magnetic field $\vec{B}$ to this rest frame, show that in the presence of non-zero resistivity, the induction equation of MHD becomes

$$\frac{\partial\vec{B}}{\partial t} + (\vec{u}\cdot\nabla)\vec{B} = (\vec{B}\cdot\nabla)\vec{u} - \vec{B}(\nabla\cdot\vec{u}) + \frac{\eta}{\mu_0}\nabla^2\vec{B}.$$

In this expression $\vec{u}$ and $\vec{B}$ are the bulk velocity of the fluid and the magnetic field in the lab frame respectively. Interpret each of the terms in this expression, and state the physical assumptions made during the derivation of this equation.

For the remainder of this question we shall consider global quantities found by integrating over the volume $\mathcal{V}$ containing the entire fluid. You may assume that *all physical quantities* ($\vec{u}, \vec{B}$, etc.) vanish on the surface bounding this volume.

(b) [5 marks] Use the induction equation to show that the total magnetic energy of the fluid evolves according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\mathcal{V}} \mathcal{E}_B\,\mathrm{d}^3V = \iiint_{\mathcal{V}} (\vec{u}\times\vec{B})\cdot\vec{J}\,\mathrm{d}^3V - \frac{\eta}{\mu_0^2}\iiint_{\mathcal{V}} |\vec{\nabla}\times\vec{B}|^2\,\mathrm{d}^3V,$$

where $\mathcal{E}_B \equiv B^2/2\mu_0$. Give a physical interpretation of both terms on the right-hand side, and state the overall effect that a non-zero resistivity has on the magnetic energy.

(c) [5 marks] Show that the total Helicity of the fluid evolves according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\mathcal{V}} \vec{A}\cdot\vec{B}\,\mathrm{d}^3V = -\frac{2\eta}{\mu_0}\iiint_{\mathcal{V}} \vec{B}\cdot(\vec{\nabla}\times\vec{B})\,\mathrm{d}^3V,$$

where $\vec{B}$ is related to the vector potential $\vec{A}$ by $\vec{B} = \nabla\times\vec{A}$.

(d) [5 marks] The effects of a non-zero resistivity become important when the magnetic field develops structures on small scales. Using the above two results for the magnetic energy and the Helicity, discuss which quantity (Helicity or magnetic energy) is more affected by resistivity in the limit of small but non-zero resistivity and small-scale structure in the magnetic field.

2. **Complex Fluids**

This is a question about Stokes flow in an incompressible fluid with constant viscosity μ.

(a) [4 marks] Consider a rigid body B that moves with velocity $\mathbf{U}$ through a fluid that is at rest at infinity. Show that the force on the body is

$$\mathbf{F} = \int_S \boldsymbol{\sigma} \cdot \mathbf{n} \, \mathrm{d}S,$$

where $\boldsymbol{\sigma}$ is the stress tensor in the fluid and S is a large spherical surface around the body with outward unit normal $\mathbf{n}$. What does this imply about the behaviour of the pressure, velocity, and strain rate far from the body?

(b) [7 marks] Consider a second flow in which the same body B moves with velocity $\mathbf{U}'$ through fluid that is at rest at infinity. The force on the body in this flow is $\mathbf{F}'$. Show that

$$\mathbf{F} \cdot \mathbf{U}' = \mathbf{F}' \cdot \mathbf{U}.$$

By considering a third flow in which the body moves with velocity $\mathbf{U} + \mathbf{U}'$ show that the force on the body must be a linear function of its velocity. Show further that

$$\mathbf{F} = -\mathsf{A} \cdot \mathbf{U},$$

where A is a symmetric tensor.

(c) [4 marks] Consider a rigid object comprising a sphere of radius a_1 with position $\mathbf{r}_1$ and a sphere of radius a_2 with position $\mathbf{r}_2$. The two spheres are joined by a rigid rod whose diameter is much smaller than $\min\{a_1, a_2\}$ and whose length is much longer than $\max\{a_1, a_2\}$. Find the tensor A for this object.

(d) [5 marks] Show that the object from part (c) has a hydrodynamic centre with position $\mathbf{r}_c$ such that the torque about $\mathbf{r}_c$ is zero when the object is held stationary in an oncoming flow that approaches a uniform stream at infinity.

(e) [5 marks] Now suppose that the object from part (c) is made to rotate with angular velocity $\boldsymbol{\Omega}$ about an axis through $\mathbf{r}_c$ in a fluid at rest at infinity. Find the force on the object. What general result does this illustrate?

[*Hint: The tensor* A *in part (b) is* $6\pi\mu a\mathsf{I}$ *for a sphere of radius* a*, where* I *is the identity tensor.*]