A15090W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

GROUPS AND REPRESENTATIONS

Hilary Term 2024

FRIDAY, 12TH JANUARY 2024, 09:30 am to 12:30 pm

You should submit answers to ***three*** *out of the four questions.*

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. (a) [4 marks] Define the terms 'representation' and 'reducible representation' of a group. For a finite group, define 'character' and briefly explain why characters are useful in the context of finite group representation theory.

   (b) [3 marks] Consider the symmetric group S_3 of permutations of three objects (taken to be $\{1,2,3\}$) and the transpositions τ_i, for $i = 1,2,3$, which leave i invariant and swap the other two numbers. Write all permutations in S_3 in terms of the transpositions τ_i and write down the conjugacy classes of S_3.

   (c) [8 marks] Determine the number of irreducible S_3 representations and their dimensions. Write down the character table of S_3. Consider the tensor products of all pairs of irreducible representations and work out their Clebsch-Gordan decomposition.

   (d) [4 marks] A two-dimensional representation $R : S_3 \to \mathrm{GL}(\mathbb{C}^2)$ is defined by

$$R(\tau_1) = \begin{pmatrix} 0 & \alpha^{-1} \\ \alpha & 0 \end{pmatrix}, \qquad R(\tau_2) = \begin{pmatrix} 0 & \alpha \\ \alpha^{-1} & 0 \end{pmatrix}, \qquad R(\tau_3) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

   where $\alpha = \exp(2\pi i/3)$. Compute the character of R and show that R is irreducible.

   (e) [6 marks] A doublet of (complex-valued) scalar fields $\phi = (\phi_1, \phi_2)^T$ transforms under the two-dimensional S_3 representation from part (d). If $V(\phi)$ is an S_3 invariant scalar potential with at most quartic terms, which powers of the fields ϕ_a, $a = 1,2$, can arise in $V(\phi)$? Write down two invariant terms explicitly. Now assume instead that ϕ transforms in the fundamental representation of $SU(2)$ and that $V(\phi)$ is required to be $SU(2)$ invariant. Which powers of the fields (up to quartic order) can arise in $V(\phi)$ in this case. Compare the result with the one for S_3.

2. (a) [6 marks] Consider the group $SU(4)$ of 4×4 special unitary matrices. Work out the Lie algebra of $SU(4)$ and its dimension. What is the Cartan sub-algebra and the rank of this Lie algebra? Write down a simple basis for the Cartan sub-algebra.

   (b) [4 marks] For the fundamental representation, $\mathbf{4}$, of $SU(4)$, find the weights of the standard unit vectors $\mathbf{e}_i$, where $i = 1, \ldots, 4$, in $\mathbb{C}^4$. What are the weights of the complex conjugate of the fundamental representation, $\bar{\mathbf{4}}$?

   (c) [6 marks] Using Young tableaux, work out the irreducible representations in $\mathbf{4} \otimes \mathbf{4}$ and $\mathbf{4} \otimes \bar{\mathbf{4}}$.

   (d) [6 marks] Consider the $SU(3)$ sub-group of $SU(4)$ defined by the embedding

$$U = \begin{pmatrix} U_3 & 0 \\ 0 & 1 \end{pmatrix},$$

   where $U_3 \in SU(3)$. How do the representations $\mathbf{4}$, $\bar{\mathbf{4}}$ and $\mathbf{4} \otimes \bar{\mathbf{4}}$ branch under this $SU(3)$ sub-group?

   (e) [3 marks] Consider an enlarged quark model with four quarks transforming in the fundamental representation of SU(4) (and the four anti-quarks of course transforming in the complex conjugate of the fundamental of SU(4)). Which mesons do you expect in such a model and how does this compare to the standard three quark model? Why is a four quark model of this kind not normally considered?

3. (a) [3 marks] Consider $\mathbb{R}^6$ with coordinates x_k, where $k = 1, \ldots, 6$, and the Euclidean metric

$$g = \sum_{k=1}^{6} dx_k^2 \ . \tag{1}$$

Show that the sub-group of $\mathrm{GL}(\mathbb{R}^6)$ which leaves this metric invariant is O(6).

(b) [6 marks] Re-write the metric in Eq. (1) in terms of complex coordinates $z_k = x_k + i x_{k+3}$, where $k = 1, 2, 3$, and their complex conjugates and show that this metric is left invariant by matrices $U \in \mathrm{SU}(3)$ acting as $z_k \mapsto \sum_j U_{kj} z_j$. Use these results to write down an explicit embedding $\mathrm{SU}(3) \subset \mathrm{SO}(6)$.

(c) [4 marks] Given the embedding from part (b), how does the fundamental representation $\mathbf{6}_{\mathrm{SO}(6)}$ branch under SU(3)? With the obvious embedding $\mathrm{SO}(6) \subset \mathrm{SO}(7)$ and the embedding $\mathrm{SU}(3) \subset \mathrm{SO}(6)$ from part (b), how does the fundamental $\mathbf{7}_{\mathrm{SO}(7)}$ branch under SU(3)?

(d) [6 marks] Now consider $\mathbb{R}^7$ with coordinates x_k, where $k = 1, \ldots, 7$, and the three-form ϕ defined by

$$\phi = dx_{147} + dx_{257} + dx_{367} + dx_{123} - dx_{156} + dx_{246} - dx_{345} =: \frac{1}{6} \sum_{k,l,m} \varphi_{klm} dx_k \wedge dx_l \wedge dx_m \ , \tag{2}$$

where the shorthand $dx_{klm} := dx_k \wedge dx_l \wedge dx_m$ has been used and φ_{klm} is a totally anti-symmetric tensor whose values are defined by Eq. (2). [Hint: The relevant rules for calculating with the wedge product are linearity, for example, $(a\, dx_1 + b\, dx_2) \wedge dx_3 = a\, dx_1 \wedge dx_3 + b\, dx_2 \wedge dx_3$ for $a, b \in \mathbb{R}$, and anti-symmetry, for example $dx_1 \wedge dx_2 = -dx_2 \wedge dx_1$.] Show that the set of matrices $P \in \mathrm{SO}(7)$ which leaves ϕ invariant forms a group. [Hint: Invariance of ϕ under P can be expressed by the equation $\sum_{n,p,q} P_{kn} P_{lp} P_{mq} \varphi_{npq} = \varphi_{klm}$.] Show that the Lie algebra of this group consists of real 7×7 matrices T which satisfy

$$\sum_n (T_{kn} \varphi_{nlm} + T_{ln} \varphi_{knm} + T_{mn} \varphi_{kln}) = 0 \ .$$

[Note: It turns out this is actually the Lie algebra G_2 and the associated group constructed above is, by abuse of notation, also often called G_2.]

(e) [6 marks] Show that the three-form ϕ in Eq. (2) can be written as

$$\phi = \omega \wedge dy + \mathrm{Re}(\Omega) \ ,$$

where

$$\omega = \frac{i}{2} \sum_{k=1}^{3} dz_k \wedge d\bar{z}_k \ , \qquad \Omega = dz_1 \wedge dz_2 \wedge dz_3 \ , \qquad dy = dx_7$$

and the complex coordinates z_1, z_2, z_3 are related to the real coordinates $x_1, \ldots, x_6$ as in part (b). Use this result to show that SU(3) is a sub-group of G_2. How does the representation $\mathbf{7}_{G_2}$ (which is induced by the fundamental representation of SO(7)) branch under this SU(3) sub-group?

4. (a) [2 marks] Consider the groups $SU(n)$, $U(n)$ and $U(1) \times SU(n) = \{(z, U) \,|\, z \in U(1)\,,\ U \in SU(n)\}$. Show that the map $f : U(1) \times SU(n) \to U(n)$ defined by $f((z, U)) := zU$ is a group homomorphism.

   (b) [4 marks] Use the homomorphism from part (a) to show that

$$U(n) \cong \frac{U(1) \times SU(n)}{\mathbb{Z}_n} \qquad (3)$$

   and specify the explicit form of the sub-group $\mathbb{Z}_n$.

   (c) [8 marks] For the group $SU(6)$, write down highest weight Dynkin labels, Young tableaux and associated tensors for the fundamental representation, the complex conjugate of the fundamental representation, the rank two symmetric and rank two anti-symmetric tensors of the fundamental representation, and the adjoint representation. What are the dimensions of these representations?

   (d) [7 marks] Embed $SU(5)$ into $SU(6)$ via

$$U \mapsto \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}\,,$$

   where $U \in SU(5)$. Given this embedding, how do the $SU(6)$ representations from part (c) branch into $SU(5)$ representations?

   (e) [4 marks] In an SU(5) GUT theory, a single standard model family of quarks and leptons is contained in the SU(5) representation $\bar{\mathbf{5}} \oplus \mathbf{10}$. Suppose you want to construct a GUT theory based on the group SU(6). Which SU(6) representation should be selected to contain one family of quarks and leptons? Given this choice, which additional SU(5) multiplets arise?