A15090H1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

Groups and Representations

HILARY TERM 2025

TUESDAY 14 January 09.30 am - 12.30 pm

This exam paper consists of four questions, each marked out of 25. You may attempt as many questions as you like, but only the best three will count for the total mark. You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. (a) [5 marks] Define the terms 'representation' and 'irreducible representation' of a group. For a finite group, provide three important properties of characters of representations.

(b) [3 marks] For a representation $R : G \to \mathrm{GL}(V)$ of a finite group G consider the linear map $P : V \to V$ defined by $P = \frac{1}{|G|}\sum_{g\in G} R(g)$. Show that $P(v)$ is a singlet under G for all vectors $v \in V$.

(c) [7 marks] Consider the group G generated by the matrices

$$g_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^* \end{pmatrix}, \qquad g_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{1}$$

where $\alpha = e^{2\pi i/3}$. Show that this group has order 6, that it is non-Abelian and find its conjugacy classes. How many irreducible representations (over complex vector spaces) does this group have and what are their dimensions?

(d) [5 marks] For the group G in part (c), find all irreducible representations (over complex vectors spaces) and write down the character table. Work out the Clebsch-Gordan decompositions for the tensor products $R \otimes \tilde{R}$ between all pairs of irreducible representations R and $\tilde{R}$.

(e) [5 marks] Consider a Yukawa coupling of the form

$$\sum_{i,j=1,2,3} \lambda_{ij} H \bar{\psi}_L^i \psi_R^j\,, \tag{2}$$

where ψ_L^i and ψ_R^i are left- and right-handed fermions, H is a complex scalar and the λ_{ij} form a 3×3 matrix λ of coupling constants. For the group from part (c), assume that $(\bar{\psi}_L^1, \bar{\psi}_L^2)$ and (ψ_R^1, ψ_R^2) transform in a two-dimensional irreducible representation, that $\bar{\psi}_L^3$ and H transform in a non-trivial one-dimensional representation and ψ_R^3 is a singlet. Find the most general form of the matrix λ so that the Yukawa terms in Eq. (2) are G-invariant.

2. (a) [4 marks] Work out the Lie algebra of SO(N) and write down a simple basis for this algebra. What are its dimension and its rank?

(b) [7 marks] A scalar field ϕ takes values in $\mathbb{R}^N$ and transforms as a fundamental of SO(N). Consider an SO(N)-invariant field theory for ϕ and suppose ϕ develops a vacuum expectation value. To which group does SO(N) spontaneously break in this case? Next, consider an SO(N)-invariant field theory for a scalar field ϕ which transforms as a second rank symmetric tensor of SO(N). What is the unbroken group now if ϕ develops a vacuum expectation value?

(c) [7 marks] Write down the Dynkin diagrams for the complex Lie algebras which are associated to SO(N). Work out the values of the quadratic Casimir and the index for the fundamental representation, as a function of N. [Hint: For the even case, $N = 2n$, the metric tensor $G(D_n)$ satisfies $(1, 0, 0, \ldots)G(D_n) = \frac{1}{2}(2, \ldots, 2, 1, 1)$ and for the odd case, $N = 2n + 1$, it satisfies $(1, 0, 0, \ldots)G(B_n) = \frac{1}{2}(2, 2, \ldots, 2, 1)$.]

(d) [7 marks] The one-loop β-function for a gauge theory with Weyl fermions in the representation r_W is given by

$$\beta(g) = -\frac{1}{16\pi^2}\left[\frac{11}{3}c(\text{adj}) - \frac{2}{3}c(r_W)\right] g^3 \ , \tag{1}$$

where $c(\text{adj})$ is the index of the adjoint representation and $c(r_W)$ is the index of the representation r_W. Suppose we have an SO(8) gauge theory with k Weyl fermions, each of which transforms in the fundamental of SO(8). Work out the β function in terms of k. For which values of k is the theory asymptotically free (as judged by the given beta-function (1))? You can use the metric tensor for D_4 given by

$$G(D_4) = \frac{1}{2}\begin{pmatrix} 2 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix} .$$

3. (a) [6 marks] Consider the Lorentz group in 1+1 (one time and one spatial) dimension and show that $\gamma_0 = -i\sigma_2$ and $\gamma_1 = \sigma_1$ are a viable choice of gamma matrices (where σ_i with $i = 1, 2, 3$ are the Pauli matrices). Write a Dirac spinor in $1 + 1$ dimensions as

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} , \tag{1}$$

where $\psi_\pm$ are in general complex. How does ψ transform under infinitesimal Lorentz transformations? [Hint: The Pauli matrices are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \qquad]$$

(b) [6 marks] Define left- and right-handed Weyl spinors in $1 + 1$ dimensions and write them in terms of the quantities in Eq. (1).

(c) [7 marks] Show that Majorana and Majorana-Weyl spinors can be defined in $1 + 1$ dimensions and specify their explicit form in terms of the quantities in Eq. (1). How many real degrees of freedom does each of the spinors discussed so far have?

(d) [6 marks] Write down a Lorentz-invariant mass term for a $1 + 1$-dimensional Dirac spinor and explicitly verify Lorentz-invariance by applying an infinitesimal transformation.

4. (a) [3 marks] Write down the Dynkin diagram and the Cartan matrix for A_5.

   (b) [8 marks] Construct the weight systems of the complex conjugate of the fundamental representation and the second rank anti-symmetric tensor of A_5.

   (c) [7 marks] The algebra A_4 is embedded into A_5 in the standard way and the projection matrix $P(A_4 \subset A_5)$ acts by deleting the last entry of the A_5 Dynkin label. Using the weight systems, find the branching under A_4 of the A_5 representations from part (b).

   (d) [7 marks] Discuss the results so far from the point of view of a standard $SU(5)$ grand unified theory. Do the $SU(6) \sim A_5$ multiplets from part (b) contain the required multiplets for an $SU(5)$ grand unified theory? Which other $SU(5)$ multiplets arise? Discuss possible physical interpretations for these additional multiplets.