A15089W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

KINETIC THEORY

Hilary Term 2024

THURSDAY, 11TH JANUARY 2024, 09:30 am to 12:30 pm

You should submit answers to all three questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. Consider a Hamiltonian system of N indistinguishable particles of unit mass interacting through a pairwise potential ϕ. Any function F of the particle positions $\mathbf{x}_i$, velocities $\mathbf{v}_i$, and time evolves according to

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial t} + \{F, H\},$$

where

$$H = \frac{1}{2}\sum_{i=1}^{N} |\mathbf{v}_i|^2 + \sum_{1\leqslant i<j\leqslant N} \phi(|\mathbf{x}_i - \mathbf{x}_j|), \quad \{A, B\} = \sum_{i=1}^{N}\left(\frac{\partial A}{\partial \mathbf{x}_i}\cdot\frac{\partial B}{\partial \mathbf{v}_i} - \frac{\partial B}{\partial \mathbf{x}_i}\cdot\frac{\partial A}{\partial \mathbf{v}_i}\right).$$

(a) [8 marks] By separating the Hamiltonian into a sum of three terms, or otherwise, show that the one-particle distribution function $f(\mathbf{x}, \mathbf{v}, t)$ evolves according to

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f = \int \mathrm{d}\mathbf{v}_\star \int \mathrm{d}\mathbf{x}_\star \, \frac{\partial f_2}{\partial \mathbf{v}} \cdot \frac{\partial \phi(|\mathbf{x} - \mathbf{x}_\star|)}{\partial \mathbf{x}},$$

where $f_2(\mathbf{x}, \mathbf{v}, \mathbf{x}_\star, \mathbf{v}_\star, t)$ is the two-particle distribution function.

(b) [4 marks] Derive an evolution equation for the fluid mass density $\rho(\mathbf{x}, t) = \int \mathrm{d}\mathbf{v} f(\mathbf{x}, \mathbf{v}, t)$, and show that the fluid momentum $\rho(\mathbf{x}, t)\mathbf{u}(\mathbf{x}, t) = \int \mathrm{d}\mathbf{v}\, \mathbf{v} f(\mathbf{x}, \mathbf{v}, t)$ evolves according to

$$\frac{\partial(\rho\mathbf{u})}{\partial t} + \nabla\cdot(\rho\mathbf{u}\mathbf{u} + \mathsf{P}) = -\int \mathrm{d}\mathbf{x}_\star \, n_2(\mathbf{x}, \mathbf{x}_\star, t)\, \frac{\partial \phi(|\mathbf{x} - \mathbf{x}_\star|)}{\partial \mathbf{x}}, \qquad (\star)$$

where the two-particle number density is

$$n_2(\mathbf{x}, \mathbf{x}_\star, t) = \int \mathrm{d}\mathbf{v} \int \mathrm{d}\mathbf{v}_\star \, f_2(\mathbf{x}, \mathbf{v}, \mathbf{x}_\star, \mathbf{v}_\star, t).$$

Give an expression for the pressure tensor P in terms of f.

(c) [8 marks] Show that $\partial n_2/\partial s = \mathbf{R}\cdot\partial n_2/\partial\mathbf{x}$ for the two-particle number density

$$n_2(\mathbf{x} + (s-1)\mathbf{R}, \mathbf{x} + s\mathbf{R}, t).$$

Hence verify that the momentum equation $(\star)$ can be written in the conservation form

$$\frac{\partial(\rho\mathbf{u})}{\partial t} + \nabla\cdot(\rho\mathbf{u}\mathbf{u} + \mathsf{P} + \mathsf{P}^\phi) = 0,$$

using the inter-particle pressure tensor P^ϕ with components

$$\mathsf{P}^\phi_{ij}(\mathbf{x}, t) = -\frac{1}{2}\int \mathrm{d}\mathbf{R}\, R_i R_j \frac{1}{R}\frac{\mathrm{d}\phi}{\mathrm{d}R}\int_0^1 \mathrm{d}s\, n_2(\mathbf{x} + (s-1)\mathbf{R}, \mathbf{x} + s\mathbf{R}, t),$$

where $R = |\mathbf{R}|$.

(d) [5 marks] The two-particle correlation function g_2 is defined by writing

$$n_2(\mathbf{x}, \mathbf{x}_\star, t) = \rho(\mathbf{x}, t)\rho(\mathbf{x}_\star, t) g_2(\mathbf{x}, \mathbf{x}_\star - \mathbf{x}, t).$$

Suppose that the system is in a spatially homogeneous and isotropic state. Show that the total pressure tensor is $\mathsf{P} + \mathsf{P}^\phi = p\mathsf{I}$, where I is the identity tensor, and

$$p = \rho\theta - \frac{2\pi}{3}\rho^2 \int_0^\infty R^3 \frac{\mathrm{d}\phi}{\mathrm{d}R}\, g_2(R, t)\, \mathrm{d}R.$$

Give an expression for θ in terms of f, and explain why g_2 can be written in this form. What is the sign of the second term for a repulsive potential?

2. Consider a one-dimensional plasma in which the electron distribution is $f(t,x,v) = f_0(t,v) + \delta f(t,x,v)$, where f_0 is the mean of f over space (over x) and over the time scales associated with the electric fluctuations. Let the electrostatic potential of the latter satisfy

$$\varphi(t,x) = \sum_k \varphi_k(t) e^{ikx}, \qquad \varphi_k = \chi_k(t) - \frac{4\pi e}{k^2} \int_{-\infty}^{+\infty} \mathrm{d}v\, \delta f_k(t,v). \tag{1}$$

Here $-e$ is the electron charge, δf_k is the Fourier transform of δf, defined in the same way as the Fourier transform φ_k of φ, and $\chi_k(t)$ is an externally imposed random noise, white in time: $\langle \chi_k(t)\chi_k^*(t')\rangle = 2A_k\delta(t-t')$, where the noise spectrum A_k is independent of time. Any perturbations of the ion distribution are to be neglected.

(a) [6 marks] Assuming $\delta f_k(t=0,v) = 0$ and using the quasilinear approximation, show that

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v}\left[D(v)\frac{\partial f_0}{\partial v}\right], \quad D(v) = \frac{e^2}{m_e^2}\sum_k \frac{k^2 A_k}{2\pi^2}\int \mathrm{d}p \int \mathrm{d}p'^* \frac{e^{(p+p'^*)t}}{(p+ikv)(p+p'^*)\epsilon(p,k)\epsilon(p',k)^*}, \tag{2}$$

where m_e is the electron mass, the integration contours in p and p' are ones associated with the inverse Laplace transform, viz., $p, p' \in (-i\infty+\sigma, +i\infty+\sigma)$ with a real constant $\sigma > 0$ to the right of any poles in the complex p plane,

$$\epsilon(p,k) = 1 - \frac{\omega_{\mathrm{pe}}^2}{k^2}\frac{1}{n_e}\int_{C_\mathrm{L}} \mathrm{d}v' \,\frac{1}{v' - ip/k}\frac{\partial f_0}{\partial v'} \approx 1 - \frac{\omega_{\mathrm{pe}}^2}{\omega^2} - i\frac{\omega_{\mathrm{pe}}^2}{k^2}\frac{\pi}{n_e} f_0'\left(\frac{\omega}{k}\right) \tag{3}$$

is the dielectric function, $\omega_{\mathrm{pe}} = (4\pi e^2 n_e/m_e)^{1/2}$ is the electron plasma frequency, n_e is the mean electron density, C_L is the Landau contour (you need not prove that this is the right contour to use), and the second expression (which you need not derive but which will be useful to you later) is an approximation valid for $p = -i\omega + \gamma$, $\gamma \ll kv_{\mathrm{the}} \ll \omega$, v_{the} being the characteristic thermal width of the electron distribution.

(b) [5 marks] Assume that f_0 is a stable distribution and hence argue that you are allowed to take $\sigma \to +0$. Show that, if we wait long enough for any damped perturbations to decay, the quasilinear diffusion coefficient is

$$D(v) = \frac{e^2}{m_e^2}\sum_k \frac{k^2 A_k}{|\epsilon(-ikv,k)|^2}. \tag{4}$$

(c) [5 marks] Consider a noise spectrum A_k concentrated at wavenumbers $k \gg \lambda_{\mathrm{De}}^{-1}$, where $\lambda_{\mathrm{De}} = v_{\mathrm{the}}/\sqrt{2}\omega_{\mathrm{pe}}$ is the electron Debye length. Show that the rate of growth of the total kinetic energy $\mathscr{K}$ per unit volume of the electrons (i.e., the heating rate) is

$$\frac{\mathrm{d}\mathscr{K}}{\mathrm{d}t} = \frac{e^2 n_e}{m_e}\sum_k k^2 A_k. \tag{5}$$

(d) [5 marks] Now consider A_k concentrated at $k \ll \lambda_{\mathrm{De}}^{-1}$. Prove that the heating rate is the same, even though a different approximation for the dielectric function has to be used. In this derivation, you may use the formula

$$\frac{1}{(kv-\omega)^2+\gamma^2} \approx \frac{\pi\delta(kv-\omega)}{|\gamma|} \quad \text{if} \quad \gamma \ll kv \sim \omega. \tag{6}$$

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(e) [4 marks] The heating in one of the limits considered above is called "resonant heating" and in the other, "stochastic heating". Which one, in your view, is which and why? Explain the physical mechanism in each case.

3. (a) [3 marks] Give two reasons why the kinetic theory of stellar sytems is usually formulated in angle-action variables $(\boldsymbol{\theta}, \mathbf{J})$ rather than position and velocity $(\mathbf{x}, \mathbf{v})$. Let $f(\boldsymbol{\theta}, \mathbf{J}, t)$ be the distribution function (DF) of an ensemble of equal mass stars, normalized such that the total mass in stars is $\int \mathrm{d}\boldsymbol{\theta}\mathrm{d}\mathbf{J} f = M$. Assuming the Hamiltonian governing individual stellar motions is some function $H(\boldsymbol{\theta}, \mathbf{J}, t)$, write down the collisionless Boltzmann (or Vlasov) equation governing the evolution of f.

(b) [1 mark] Let $f(\boldsymbol{\theta}, \mathbf{J}, t) = f_0(\mathbf{J}) + \delta f(\boldsymbol{\theta}, \mathbf{J}, t)$, where $f_0(\mathbf{J})$ is an unperturbed DF, and let

$$H(\boldsymbol{\theta}, \mathbf{J}, t) = H_0(\mathbf{J}) + \delta\Phi^{\mathrm{tot}}(\boldsymbol{\theta}, \mathbf{J}, t). \tag{7}$$

By linearizing the Vlasov equation under the assumption that the fluctuations δf, $\delta\Phi^{\mathrm{tot}}$ are small, write down the equation determining the time evolution of δf.

(c) [4 marks] Define the Fourier-Laplace transform of an arbitrary smooth function g as

$$\widetilde{g}_{\mathbf{n}}(\mathbf{J}, \omega) = \int_0^\infty \mathrm{d}t \exp(i\omega t) \int \frac{\mathrm{d}\boldsymbol{\theta}}{(2\pi)^3} \exp(-i\mathbf{n}\cdot\boldsymbol{\theta})\, g(\boldsymbol{\theta}, \mathbf{J}, t), \tag{8}$$

where $\mathrm{Im}\,\omega > 0$ is large enough for the integral to converge, and $\mathbf{n} \in \mathbb{Z}^3$. Using the result of part (b), and assuming that $\delta f = 0$ at $t = 0$, show that

$$\widetilde{\delta f}_{\mathbf{n}}(\mathbf{J}, \omega) = -\mathbf{n}\cdot\frac{\partial f_0}{\partial \mathbf{J}} \frac{\widetilde{\delta\Phi}^{\mathrm{tot}}_{\mathbf{n}}(\mathbf{J}, \omega)}{\omega - \mathbf{n}\cdot\boldsymbol{\Omega}(\mathbf{J})}, \tag{9}$$

where you should define the frequency vector $\boldsymbol{\Omega}(\mathbf{J})$.

(d) [9 marks] Consider a fictitious, homogeneous stellar system, placed within a periodic 3D cube of side-length L, and with zero mean potential $\Phi_0 = 0$. Then one has

$$\boldsymbol{\theta} = \frac{2\pi}{L}\mathbf{x}, \qquad \mathbf{J} = \frac{L}{2\pi}\mathbf{v}, \qquad \boldsymbol{\Omega} = \frac{2\pi}{L}\mathbf{v}. \tag{10}$$

Assume further that the total potential fluctuation $\delta\Phi^{\mathrm{tot}} = \delta\Phi + \delta\Phi^{\mathrm{ext}}$, where $\delta\Phi$ is self-consistently generated from δf, and $\delta\Phi^{\mathrm{ext}}$ is externally imposed. Using equation (9), show that for this system

$$\widetilde{\delta\Phi}^{\mathrm{tot}}_{\mathbf{k}}(\omega) = \frac{\widetilde{\delta\Phi}^{\mathrm{ext}}_{\mathbf{k}}(\omega)}{\epsilon_{\mathbf{k}}(\omega)}, \tag{11}$$

where

$$\epsilon_{\mathbf{k}}(\omega) = 1 + \frac{4\pi G}{k^2} \int \mathrm{d}\mathbf{v}\, \frac{1}{\mathbf{k}\cdot\mathbf{v} - \omega}\, \mathbf{k}\cdot\frac{\partial f_0}{\partial \mathbf{v}}, \tag{12}$$

and $\mathbf{k} = 2\pi\mathbf{n}/L$. How can this result be analytically continued to the entire complex ω plane? What do the zeroes of $\epsilon_{\mathbf{k}}(\omega)$ correspond to?

(e) [3 marks] Assume that the external perturbation is due to a point mass m_{p}, which is introduced at $t = 0$ and thereafter moves on the straight line trajectory $\mathbf{x} = \mathbf{x}_{\mathrm{p}} + \mathbf{v}_{\mathrm{p}}t$. Show that in this case,

$$\widetilde{\delta\Phi}^{\mathrm{ext}}_{\mathbf{k}}(\omega) = -\frac{4\pi G m_{\mathrm{p}}}{L^3 k^2} \exp(-i\mathbf{k}\cdot\mathbf{x}_{\mathrm{p}}) \frac{1}{i(\mathbf{k}\cdot\mathbf{v}_{\mathrm{p}} - \omega)}. \tag{13}$$

(f) [5 marks] Assume that f_0 is stable. Using the residue theorem, show that the time-asymptotic linear response of the system's total potential satisfies

$$\delta\Phi^{\mathrm{tot}}_{\mathbf{k}}(t) = \frac{\delta\Phi^{\mathrm{ext}}_{\mathbf{k}}(t)}{\epsilon_{\mathbf{k}}(\mathbf{k}\cdot\mathbf{v}_{\mathrm{p}})}. \tag{14}$$

Interpret this result physically. How long do we have to wait for this result to become valid?