A15089H1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

Kinetic Theory

HILARY TERM 2025

THURSDAY 16 January, 9.30am - 12.30pm

This exam paper consists of three questions each marked out of 25. You should submit answers to all three questions. You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. Consider a Hamiltonian system of N indistinguishable particles of unit mass subject to an external potential U and interacting through a pairwise potential ϕ. Any function F of the particle positions $\mathbf{x}_i$ and velocities $\mathbf{v}_i$ evolves according to

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \{F, H\},$$

where

$$H = \sum_{i=1}^{N} \left(\frac{1}{2}|\mathbf{v}_i|^2 + U(\mathbf{x}_i) \right) + \sum_{1 \leqslant i < j \leqslant N} \phi(|\mathbf{x}_i - \mathbf{x}_j|), \quad \{A, B\} = \sum_{i=1}^{N} \left(\frac{\partial A}{\partial \mathbf{x}_i} \cdot \frac{\partial B}{\partial \mathbf{v}_i} - \frac{\partial B}{\partial \mathbf{x}_i} \cdot \frac{\partial A}{\partial \mathbf{v}_i} \right).$$

(a) [8 marks] By separating the Hamiltonian into a sum of three terms, or otherwise, show that the one-particle distribution function $f(\mathbf{x}, \mathbf{v}, t)$ evolves according to

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f - \nabla U \cdot \nabla_{\mathbf{v}} f = \int \mathrm{d}\mathbf{v}_\star \int \mathrm{d}\mathbf{x}_\star \, \nabla \phi(|\mathbf{x} - \mathbf{x}_\star|) \cdot \nabla_{\mathbf{v}} f_2,$$

where $f_2(\mathbf{x}, \mathbf{v}, \mathbf{x}_\star, \mathbf{v}_\star, t)$ is the two-particle distribution function, ∇ is the gradient with respect to $\mathbf{x}$, and $\nabla_{\mathbf{v}}$ is the gradient with respect to $\mathbf{v}$.

(b) [6 marks] Briefly describe the approximations that allow this evolution equation for f to be approximated by

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f - \nabla U \cdot \nabla_{\mathbf{v}} f = \int \mathrm{d}\mathbf{v}_\star \int \mathrm{d}\theta \int \mathrm{d}\varphi \; B(|\mathbf{v} - \mathbf{v}_\star|, \theta) \left(f' f'_\star - f f_\star \right),$$

where the integration is over a unit hemisphere in θ and φ coordinates, $B(|\mathbf{v}-\mathbf{v}_\star|, \theta)$ is the Boltzmann scattering kernel, $f_\star = f(\mathbf{x}, \mathbf{v}_\star, t)$ and similarly for f' and $f'_\star$. Briefly describe the further approximations that lead to

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f - \nabla U \cdot \nabla_{\mathbf{v}} f = -\frac{1}{\tau} \left(f - f^{(0)} \right),$$

where

$$f^{(0)}(\mathbf{x}, \mathbf{v}, t) = \frac{\rho}{(2\pi\Theta)^{3/2}} \exp\left(-\frac{|\mathbf{v} - \mathbf{u}|^2}{2\Theta} \right).$$

Explain how ρ, $\mathbf{u}$, and Θ are determined from f, and give an interpretation of the constant τ.

(c) [4 marks] Show that the fluid momentum evolves according to an equation of the form

$$\partial_t(\rho \mathbf{u}) + \nabla \cdot \mathbf{\Pi} = \mathbf{F},$$

and give expressions for $\mathbf{\Pi}$ and $\mathbf{F}$.

(d) [7 marks] Show that the pressure tensor $\mathsf{P} = \mathbf{\Pi} - \rho \mathbf{u}\mathbf{u}$ evolves according to

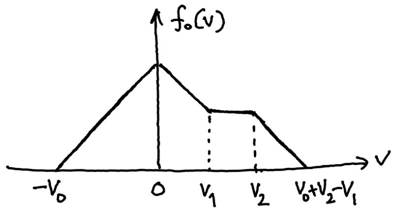
$$\partial_t P_{ij} + \partial_k \left(u_k P_{ij} + Q_{ijk} \right) + P_{ik} \frac{\partial u_j}{\partial x_k} + P_{jk} \frac{\partial u_i}{\partial x_k} = -\frac{1}{\tau} \left(P_{ij} - P_{ij}^{(0)} \right),$$

and give an expression for Q_{ijk} in terms of f.

2. Consider a one-dimensional plasma in which the equilibrium electron distribution $f_0(v)$ is as depicted in the figure below: $f_0(v) \neq 0$ only for $v \in [-v_0, v_0 + v_2 - v_1]$, with a constant slope $f_0'(v) = \pm f_0(0)/v_0$ (positive at $v < 0$, negative at $v > 0$) everywhere inside that interval except at $v \in [v_1, v_2]$, where it has a plateau with $f_0'(v) = 0$. We seek linear perturbations of this plasma that have phase velocities greatly exceeding the characteristic width of the ion distribution, so the latter's contribution to the dielectric function can be ignored:

$$\epsilon(p,k) = 1 - \frac{\omega_{\rm pe}^2}{k^2}\frac{1}{n_0}\int_{C_{\rm L}} {\rm d}v\,\frac{f_0'(v)}{v - ip/k}, \tag{1}$$

where we define $n_0 = v_0 f_0(0)$ and $\omega_{\rm pe} = (4\pi e^2 n_0/m_e)^{1/2}$, $-e$ and m_e being the electron charge and mass, respectively, and $C_{\rm L}$ is the Landau contour.



(a) [3 marks] At what values of the phase velocity $u = \omega/k$ do you expect *a priori*, from the form of (1), that completely undamped waves ($p = -i\omega$, where ω is real) might be able to exist?

(b) [5 marks] Show that the frequencies $\omega = ku$ and wavenumbers k of such undamped waves must satisfy the following dispersion relation

$$\ln\frac{u^2|v_2 - u|}{|v_0 + u||v_1 - u||v_0 + v_2 - v_1 - u|} = (k\lambda_{\rm De})^2, \tag{2}$$

where, by definition, $\lambda_{\rm De} = v_0/\omega_{\rm pe}$. You will be able to do parts (c)–(f) using (2).

(c) [3 marks] What waves exist in this plasma at short wavelengths, $k\lambda_{\rm De} \gg 1$? Why are they undamped?

(d) [5 marks] Now consider long wavelengths, $k\lambda_{\rm De} \ll 1$. Assume that $v_2 - v_1 \ll v_0, v_1$. Show that the dispersion relation has three solutions:

$$\omega \approx \pm\omega_{\rm pe}, \qquad \omega \approx k\left[v_1 + (v_2 - v_1)\frac{v_1^2}{v_0^2}\right]. \tag{3}$$

The first two are the familiar Langmuir waves (plasma oscillations) and the third resembles a sound wave, so could be called the *electron acoustic wave (EAW)*. Why does the EAW not exist in a Maxwellian plasma?

(e) [5 marks] Continue assuming $v_2 - v_1 \ll v_0, v_1$, but consider arbitrary wavenumbers $k\lambda_{\rm De}$. Derive the dispersion relation for the EAW,

$$\omega \approx k\left[v_1 + \frac{v_2 - v_1}{1 + e^{k^2\lambda_{\rm De}^2}(v_0^2 - v_1^2)/v_1^2}\right], \tag{4}$$

and explain how it relates to your previous results.

(f) [4 marks] Assembling together the results that you have derived, sketch the three branches of the dispersion relation ω vs. k for an electron plasma with a small plateau.

3. (a) [2 marks] Give two reasons why the kinetic theory of stellar sytems is usually formulated in angle-action variables $(\boldsymbol{\theta}, \mathbf{J})$ rather than position and velocity $(\mathbf{x}, \mathbf{v})$.

(b) [5 marks] Let f be the distribution function (DF) of a razor-thin disk of stars whose phase space location is determined by angle-action coordinates $\boldsymbol{\theta} = (\theta_\varphi, \theta_R)$, $\mathbf{J} = (J_\varphi, J_R)$, and whose motion is governed by the 'mean field + perturbation' Hamiltonian

$$H(\boldsymbol{\theta}, \mathbf{J}, t) = H_0(\mathbf{J}) + \delta\Phi(\boldsymbol{\theta}, \mathbf{J}, t). \tag{1}$$

Let $f(\boldsymbol{\theta}, \mathbf{J}, t) = f_0(\mathbf{J}, t) + \delta f(\boldsymbol{\theta}, \mathbf{J}, t)$, where $f_0(\mathbf{J}, t)$ is the angle-independent part of the DF. Fourier expanding the potential as $\delta\Phi = \sum_{\mathbf{k}} \delta\Phi_{\mathbf{k}}(\mathbf{J}, t) \exp(i\mathbf{k} \cdot \boldsymbol{\theta})$ and similarly for δf, where $\mathbf{k} = (k_\varphi, k_R) \in \mathbb{Z}^2$, assuming all perturbations are small, and ignoring initial conditions, show that the linear response of the DF satisfies

$$\delta f_{\mathbf{k}}(\mathbf{J}, t) = i \int_0^t \mathrm{d}t' \, \mathbf{k} \cdot \frac{\partial f_0(\mathbf{J}, t')}{\partial \mathbf{J}} \mathrm{e}^{-i\mathbf{k}\cdot\boldsymbol{\Omega}(t-t')} \delta\phi_{\mathbf{k}}(\mathbf{J}, t'). \tag{2}$$

where you should define the frequency vector $\boldsymbol{\Omega}(\mathbf{J})$.

(c) [6 marks] Define the marginalized DF of angular momenta $F_0(J_\varphi, t) \equiv 2\pi \int_0^\infty \mathrm{d}J_R \, f_0(\mathbf{J}, t)$. By first deriving an equation for $\partial f_0/\partial t$, show that to second order in small quantities,

$$\frac{\partial F_0}{\partial t} = -\frac{\partial}{\partial J_\varphi} \mathcal{Q}_0, \tag{3}$$

where the flux $\mathcal{Q}_0$ is given by

$$\mathcal{Q}_0(J_\varphi, t) = -2\pi \sum_{\mathbf{k}} k_\varphi \int_0^\infty \mathrm{d}J_R \int_0^t \mathrm{d}t' \mathbf{k} \cdot \frac{\partial f_0(\mathbf{J}, t')}{\partial \mathbf{J}} \mathrm{e}^{-i\mathbf{k}\cdot\boldsymbol{\Omega}(t-t')} \delta\phi_{\mathbf{k}}(\mathbf{J}, t') \delta\phi^*_{\mathbf{k}}(\mathbf{J}, t). \tag{4}$$

(d) [3 marks] Now specialize to a perturbation of the form

$$\delta\phi_{\mathbf{k}}(\mathbf{J}, t) = u_{\mathbf{k}}(\mathbf{J}) \mathrm{e}^{-ik_\varphi\Omega_{\mathrm{p}}t} \mathrm{e}^{-(t-t_{\mathrm{peak}})^2/(2\tau^2)}, \tag{5}$$

where the function $u_{\mathbf{k}}(\mathbf{J})$ is equal to zero unless $k_\varphi = \pm m$. Interpret what this perturbation might correspond to physically. Show that it drives a flux

$$\mathcal{Q}_0(J_\varphi, t) = -2\pi \sum_{k_\varphi = \pm m} \sum_{k_R} k_\varphi \int_0^\infty \mathrm{d}J_R \, |u_{\mathbf{k}}(\mathbf{J})|^2 \times \int_0^t \mathrm{d}t' \mathbf{k} \cdot \frac{\partial f_0(\mathbf{J}, t')}{\partial \mathbf{J}} \mathrm{e}^{-i\omega_{\mathbf{k}}(\mathbf{J})(t-t')} \mathrm{e}^{-(t-t_{\mathrm{peak}})^2/(2\tau^2)} \mathrm{e}^{-(t'-t_{\mathrm{peak}})^2/(2\tau^2)}, \tag{6}$$

where you should define the frequency $\omega_{\mathbf{k}}(\mathbf{J})$.

(e) [7 marks] We will now calculate the total change to the angular momentum DF, $\Delta F_0(J_\varphi) \equiv F_0(J_\varphi, t \to \infty) - F_0(J_\varphi, 0)$. First, argue that if the perturbation is sufficiently short lived, we can replace all lower time integration limits by $-\infty$ and we can ignore the slow time evolution of $f_0(\mathbf{J}, t')$ on the right hand side of (6). Then, assuming $\omega_{\mathbf{k}}(\mathbf{J})$ does not depend on J_R, show that

$$\Delta F_0 = \frac{(8\pi^5)^{1/2} m}{\Gamma} \frac{\partial}{\partial J_\varphi} \sum_{k_R} \mathcal{R}_\Gamma(\omega_{mk_R}) \int_0^\infty \mathrm{d}J_R \, |u_{mk_R}|^2 \left(m \frac{\partial f_0}{\partial J_\varphi} + k_R \frac{\partial f_0}{\partial J_R} \right), \quad (7)$$

with $\Gamma \equiv (\sqrt{2}\tau)^{-1}$ and $\mathcal{R}_\Gamma(\omega) \equiv (\sqrt{2\pi}\Gamma)^{-1} \mathrm{e}^{-\omega^2/(2\Gamma^2)}$. You may use the identity

$$\mathrm{Re} \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{x} \mathrm{d}y \, \mathrm{e}^{-ia(x-y)} \mathrm{e}^{-(x^2+y^2)/(2\sigma^2)} = \pi\sigma^2 \mathrm{e}^{-a^2\sigma^2}. \quad (8)$$

(f) [2 marks] Give a physical interpretation of this result, focusing on the contributions from (i) $k_R = 0$ and (ii) $k_R = \pm 1$.