A15088W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

QUANTUM FIELD THEORY

Hilary Term 2024

WEDNESDAY, 10TH JANUARY 2024, 09:30 am to 12:30 pm

You should submit answers to all three questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. The quantized complex scalar field $\phi(t,\boldsymbol{x})$ and its Hamiltonian H are given by

$$\phi(t,\boldsymbol{x}) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\left(b^\dagger_{-\boldsymbol{p}}e^{iE_{\boldsymbol{p}}t} + a_{\boldsymbol{p}}e^{-iE_{\boldsymbol{p}}t}\right)e^{i\boldsymbol{p}\cdot\boldsymbol{x}},$$
$$H = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\,E_{\boldsymbol{p}}\,(a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}} + b^\dagger_{\boldsymbol{p}}b_{\boldsymbol{p}})\,,$$

where

$$\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}\right] = \left[b_{\boldsymbol{p}}, b^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})\,,$$

and all other commutators vanish.

(a) [4 marks] Show that $\phi(t,\boldsymbol{x})$ obeys the Heisenberg equation of motion

$$i\frac{\partial\phi(t,\boldsymbol{x})}{\partial t} = [\phi(t,\boldsymbol{x}), H].$$

(b) [4 marks] Write down the expression for the total momentum operator $\boldsymbol{P}$ in terms of the annihilation and creation operators. Show that

$$-i\boldsymbol{\nabla}\phi(t,\boldsymbol{x}) = [\phi(t,\boldsymbol{x}), \boldsymbol{P}]\,.$$

(c) [5 marks] The time and space translation operators are e^{-itH} and $e^{i\boldsymbol{x}\cdot\boldsymbol{P}}$ respectively. Using the results (a) and (b), show explicitly that these operators give the correct result for infinitessimal transformations.

(d) [4 marks] Assume that the results above apply (with appropriately defined $\boldsymbol{P}$ and H) for a quantized scalar field $\phi(x)$ in an interacting field theory. Let $|\Omega\rangle$ denote the Lorentz invariant vacuum and $|\psi,\boldsymbol{p}\rangle$ a scalar state with rest mass m_ψ and three momentum $\boldsymbol{p}$. Show that

$$\langle\Omega|\phi(x)|\psi,\boldsymbol{p}\rangle = e^{-ip\cdot x}\langle\Omega|\phi(0)|\psi,\boldsymbol{p}\rangle\,,$$

and give the components of the 4-vector p^μ.

(e) [8 marks] Assume that there exists a unitary Lorentz boost operator U_β such that $U_\beta|\psi,\boldsymbol{p}\rangle = |\psi,\boldsymbol{p}_\beta\rangle$, where $\boldsymbol{p}_\beta$ is the result of the boost β on $\boldsymbol{p}$, and that $\phi(x)$ is a Lorentz scalar. Show that

$$\langle\Omega|\phi(0)|\psi,\boldsymbol{p}\rangle = \langle\Omega|\phi(0)|\psi,\boldsymbol{0}\rangle\,.$$

With the same assumptions find the eigenstate $|\Psi\rangle$ of energy E^Ψ such that

$$\langle\chi,\boldsymbol{q}|\phi(x)|\psi,\boldsymbol{p}\rangle = \langle\Psi|\phi(0)|\psi,\boldsymbol{0}\rangle\,,$$

and find an expression for E^Ψ in terms of p, q and m_ψ.

2. The Dirac gamma matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \quad \mu, \nu = 0, 1, 2, 3\,, \tag{1}$$

where $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$; and $u^s(\boldsymbol{p})$ denotes the Dirac spinor normalized to $u^s(\boldsymbol{p})^\dagger u^{s'}(\boldsymbol{p}) = 2E_{\boldsymbol{p}}\delta^{ss'}$.

(a) [3 marks] Assuming (1), show that $\not{p}\not{p} = p^2$, and hence that any polynomial of $\not{p}$ must take the form $A(p^2)\not{p} + B(p^2)$.

(b) [6 marks] Define $P(\boldsymbol{p})$ by $(P(\boldsymbol{p}))_{ab} = \sum_s u^s_a(\boldsymbol{p})\overline{u}^s_b(\boldsymbol{p})$. Show that $(\not{p} - m)P(\boldsymbol{p}) = 0$, and that $\text{Tr}[P(\boldsymbol{p})\gamma^0] = 4E_{\boldsymbol{p}}$. Hence, by assuming $P(\boldsymbol{p})$ must be a polynomial in $\not{p}$, and not otherwise, show that $P(\boldsymbol{p}) = \not{p} + m$.

(c) [8 marks] The Lagrangian density for a system consisting of a real scalar and two types of Dirac fermion in four dimensions is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2 + \sum_{i=1,2}\overline{\psi}_i(i\gamma^\mu\partial_\mu - m)\psi_i - g\phi\overline{\psi}_i\psi_i\,.$$

A type 1 fermion with spin and four-momenta s_1, p_1 annihilates with a type 1 anti-fermion with spin and four-momenta s_2, p_2 producing a type 2 fermion with spin and four-momenta s_3, p_3 and a type 2 anti-fermion with spin and four-momenta s_4, p_4. Draw the tree-level Feynman diagram, including momentum and particle labelling, for this process. Using momentum space Feynman rules for Minkowski space-time, write down a formula for the matrix element $M^{s_1,s_2,s_3,s_4}_{p_1,p_2,p_3,p_4}$ corresponding to your diagram. Show that the total amplitude squared for the spin averaged scattering of unpolarised particles to a final state of any spin is given by

$$|M|^2 = g^4\left(\frac{4m^2 - s}{M^2 - s}\right)^2\,,$$

where $s = (p_1 + p_2)^2$. [You may assume that $\overline{P}(\boldsymbol{p}) = \sum_s v^s_a(\boldsymbol{p})\overline{v}^s_b(\boldsymbol{p}) = (\not{p} - m)_{ab}$.]

(d) [8 marks] A type 1 fermion with spin and four-momenta s_1, p_1 scatters off a type 1 anti-fermion with spin and four-momenta s_2, p_2 producing a type 1 fermion with spin and four-momenta s_3, p_3 and a type 1 anti-fermion with spin and four-momenta s_4, p_4. Draw the tree-level Feynman diagrams for this process. How does the matrix element transform under $p_1 \leftrightarrow -p_4$? Making use of the result from (c), and without further detailed calculation, deduce as much as you can about the total amplitude squared for the scattering of unpolarised particles to a final state of any spin.

3. This question concerns the theory of a real scalar field ϕ and a two-component fermion *in 3 dimensions* (one time and two space) whose Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2 + \overline{\psi}(i\sigma^\mu\partial_\mu - m)\psi - \frac{g_1}{2}\phi^2\overline{\psi}\psi - \frac{g_2}{4!}\phi^4 - \frac{g_3}{6!}\phi^6\,, \quad \mu = 0,1,2\,.$$

Here $\gamma^0 = \sigma_3$, $\gamma^1 = i\sigma_1$, $\gamma^2 = i\sigma_2$, σ_i are the Pauli matrices, and $\overline{\psi} \equiv \psi^\dagger\gamma^0$.

No credit will be given for answers to the following questions that do not work in 3 dimensions.

(a) [3 marks] What are the dimensions of g_1, g_2 and g_3?

(b) [8 marks] The superficial degree of divergence of an amputated Feynman graph $\Gamma(p_i)$, where $p_i \gg M, m$ are external line momenta, is defined by

$$\Gamma(\lambda p_i) = \lambda^\omega \Gamma(p_i)\,.$$

Show that the superficial degree of divergence of a graph with b external boson lines, f external fermion lines and n_i vertices with coupling g_i is given by

$$\omega = 3 - \frac{1}{2}b - f - n_2.$$

(c) [7 marks] List the correlation functions that are superficially divergent in this theory and explain why it is expected to be renormalizable.

(d) [7 marks] Assuming that $g_3 = 0$, show that to one-loop order no ϕ^6 counter-term is required. Discuss what happens at two-loop order.