A15088H1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

Quantum Field Theory

HILARY TERM 2025

WEDNESDAY 15 January, 9.30 am - 12.30 pm

This exam paper consists of three questions each marked out of 25. You should submit answers to all three questions. You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. The Lagrangian density for a fermion of mass M interacting with a boson of mass m in four space-time dimensions is given by

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu\partial_\mu - M)\psi + \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) - g\phi\overline{\psi}\gamma^5\psi\,,$$

where the Dirac gamma matrices satisfy

$$\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I} \quad \mu,\nu = 0,1,2,3\,, \tag{1}$$

with $\eta^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$, and we define $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$.

(a) [6 marks] Using the anticommutation rules (1) show that $\{\gamma^5,\gamma^\mu\} = 0$, and $\gamma^5\gamma^5 = -\mathbb{I}$. Show that $\mathrm{Tr}\,\gamma^5 = \mathrm{Tr}\,\gamma^5\gamma^\mu = 0$ [Hint: use the cyclic property of the Trace].

(b) [4 marks] Write down the most general counterterm Lagrangian, $\mathcal{L}_{ct}$, for this theory.

(c) [8 marks] What is the Feynman rule for the interaction vertex arising from $\mathcal{L}$? Show that the one loop contributions to the bosonic one- and three- point vertices vanish.

(d) [7 marks] $\mathcal{L}$ is invariant under parity $\boldsymbol{x} \to -\boldsymbol{x}$. State the corresponding transformation law for ψ (it is not required to prove this law). What is the transformation law for ϕ? Discuss the implications of this symmetry for $\mathcal{L}_{ct}$ to all orders in perturbation theory.

2. The Lagrangian density for a system in four space-time dimensions consisting of two complex scalar fields Φ and ϕ with masses M and m respectively, where $M \gg m$, is given by

$$\mathcal{L} = \partial_\mu\Phi^\dagger\partial^\mu\Phi + \partial_\mu\phi^\dagger\partial^\mu\phi - M^2\Phi^\dagger\Phi - m^2\phi^\dagger\phi - g\phi^\dagger\phi\,\Phi^\dagger\Phi - \tfrac{1}{2}h(\phi^2\Phi + h.c).$$

(a) [4 marks] What is the internal symmetry of $\mathcal{L}$ for $g \neq 0\,, h \neq 0$? What is the corresponding conserved current?

(b) [4 marks] Write down the Feynman rules, taking care to show the flow of charge.

(c) [6 marks] Draw the tree-level Feynman graphs for:

i) scattering of two ϕ particles, $\phi\phi \to \phi\phi$;

ii) scattering of a ϕ particle and a Φ particle, $\phi\Phi \to \phi\Phi$.

iii) the decay of a Φ particle, $\Phi \to \phi\phi$.

Write down the matrix elements $\mathcal{M}$ for each of these processes.

(d) [7 marks] The width for a particle of mass M decaying into two identical final state particles of mass m with three momenta $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ is given by

$$\Gamma_\Phi = \frac{1}{2M}\int\frac{d^3\boldsymbol{p}_1}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}_1}}\int\frac{d^3\boldsymbol{p}_2}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}_2}}|\mathcal{M}|^2(2\pi)^4\delta^4(p_1+p_2-P)\,,$$

where $P^\mu = (M,0,0,0)$, and $E_{\boldsymbol{p}} = +\sqrt{\boldsymbol{p}^2+m^2}$. Evaluate Γ_Φ.

(e) [4 marks] Assuming that $g = 0$, the cross-section for $\phi\phi$ scattering when $s \gg M^2$ is given by

$$\sigma(\phi\phi \to \phi\phi) = \frac{h^4}{32\pi s^3}\,.$$

What happens to the matrix element for this process when $s \simeq M^2$? Find $\sigma(\phi\phi \to \phi\phi)$ when $s \simeq M^2$.

3. The quantized Dirac field $\psi(t,\boldsymbol{x})$ and its Hamiltonian H are given by

$$\begin{aligned}\psi(t,\mathbf{x}) &= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\sum_{s=\pm}\left(e^{-ip\cdot x}a^s_{\mathbf{p}}u^s(\mathbf{p}) + e^{ip\cdot x}b^{s\dagger}_{\mathbf{p}}v^s(\mathbf{p})\right) ,\\ H &= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} E_{\boldsymbol{p}}\sum_{s=\pm}(a^{s\dagger}_{\mathbf{p}}a^s_{\mathbf{p}} + b^{s\dagger}_{\mathbf{p}}b^s_{\mathbf{p}}) ,\end{aligned}$$

where the annihilation and creation operators for particles and antiparticles satisfy the anti-commutation rules,

$$\left\{a^{s'}_{\boldsymbol{p}}, a^{s\dagger}_{\boldsymbol{q}}\right\} = \left\{b^{s'}_{\boldsymbol{p}}, b^{s\dagger}_{\boldsymbol{q}}\right\} = (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})\delta^{ss'} ,$$

with all other anticommutators vanishing.

(a) [5 marks] Show that $\psi(t,\boldsymbol{x})$ obeys the Heisenberg equation of motion

$$i\frac{\partial\psi(t,\boldsymbol{x})}{\partial t} = [\psi(t,\boldsymbol{x}), H].$$

(b) [3 marks] Show that the two particle state

$$|\boldsymbol{p}_1, s_1, \boldsymbol{p}_2, s_2\rangle = a^{s_1\dagger}_{\boldsymbol{p}_1}a^{s_2\dagger}_{\boldsymbol{p}_2}|\emptyset\rangle ,$$

where $|\emptyset\rangle$ is the Fock vacuum state, is antisymmetric under exchange of the two particles.

(c) [5 marks] Show that the n particle state

$$|\boldsymbol{p}_1, s_1, \boldsymbol{p}_2, s_2, \ldots \boldsymbol{p}_n, s_n\rangle = a^{s_1\dagger}_{\boldsymbol{p}_1}a^{s_2\dagger}_{\boldsymbol{p}_2}\ldots a^{s_n\dagger}_{\boldsymbol{p}_n}|\emptyset\rangle$$

is antisymmetric under the exchange of *any* two particles.

(d) [6 marks] Show that

$$\begin{aligned}\langle\emptyset|T\,\overline{\psi}_a(x)\,\psi_b(y)|\emptyset\rangle = \int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}\Bigg(&\theta(x^0-y^0)e^{-ip\cdot(x-y)}\sum_s\overline{v}^s_a(\boldsymbol{p})v^s_b(\boldsymbol{p})\\ &+\theta(y^0-x^0)e^{ip\cdot(x-y)}\sum_s\overline{u}^s_a(\boldsymbol{p})u^s_b(\boldsymbol{p})\Bigg) .\end{aligned}$$

(e) [6 marks] Discuss the significance and form of the result obtained in part d). What are the properties and physical role of the individual elements?